Solution 3

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Problem 1

Convert the following formulas to polish notation and reverse polish notation.

1. $P \wedge \neg R \leftrightarrow P \vee Q$
2. $\neg\neg P \vee (W \wedge R) \vee \neg Q$

Solution.

1. Polish notation:$\leftrightarrow \wedge P \neg R \vee PQ$. Reverse polish notation: $PR\neg \wedge PQ\vee \leftrightarrow$.
2. Polish notation:$\vee \vee \neg\neg P \wedge WR\neg Q$. Reverse polish notation: $P\neg\neg WR \wedge \vee Q\neg\vee$.

Problem 2

Convert the following formulas to CNF, DNF, PCNF and PDNF.

1. $P \vee \neg P$
2. $P \wedge \neg P$
3. $P \leftrightarrow (Q \rightarrow (Q \rightarrow P))$

Solution.

1. CNF: $P \vee \neg P$
   DNF: $P \vee \neg P$
   PCNF: none
   PDNF: $P \vee \neg P = \vee_{0;1}$

2. CNF: $P \wedge \neg P$
   DNF: $P \wedge \neg P$
   PCNF: $P \wedge \neg P = \wedge_{0;1}$
   PDNF: *none*

3. CNF:

$$\begin{aligned}
&P \leftrightarrow (Q \rightarrow (Q \rightarrow P)) \\
=&P \leftrightarrow (\neg Q \vee \neg Q \vee P) \\
=&P \leftrightarrow (\neg Q \vee P) \\
=&(P \rightarrow (\neg Q \vee P)) \wedge ((\neg Q \vee P) \rightarrow P) \\
=&(\neg P \vee \neg Q \vee P) \wedge ((Q \wedge \neg P) \vee P) \\
=&(P \vee Q) \wedge (P \vee \neg P) \\
=&P \vee Q
\end{aligned}$$

   DNF: $P \vee Q$
   PCNF: $P \leftrightarrow (Q \rightarrow (Q \rightarrow P)) = P \vee Q = \wedge_3$
   PDNF: $P \leftrightarrow (Q \rightarrow (Q \rightarrow P)) = \wedge_3 = \vee_{1;2;3} = (\neg P \wedge Q) \vee (P \wedge \neg Q) \vee (P \wedge Q)$

Problem 3

Prove the following formula with three proof methods, including theorem 2.8.1 in p32 of the textbook, theorem 2.8.2 in p32 of the textbook and interpretaion method in p32 of the textbook.

$$(P \wedge Q) \Rightarrow (P \rightarrow Q)$$

Solution.

1.

$$
\begin{aligned}
&(P \wedge Q) \rightarrow (P \rightarrow Q) \\
=&\neg(P \wedge Q) \vee (\neg P \vee Q) \\
=&\neg P \vee \neg Q \vee \neg P \vee Q \\
=&\neg P \vee \neg P \vee \neg Q \vee Q \\
=&T
\end{aligned}
$$

Because $(P \wedge Q) \rightarrow (P \rightarrow Q)$ is a tautology, $(P \wedge Q) \Rightarrow (P \rightarrow Q)$ is proved.

2.

$$
\begin{aligned}
&(P \wedge Q) \wedge \neg(P \rightarrow Q) \\
=&(P \wedge Q) \wedge (P \wedge \neg Q) \\
=&P \wedge Q \wedge P \wedge \neg Q \\
=&F
\end{aligned}
$$

Because $(P \wedge Q) \wedge \neg(P \rightarrow Q)$ is a contradiction, $(P \wedge Q) \Rightarrow (P \rightarrow Q)$ is proved.

3. Assuming that $P \wedge Q = T$, then $P = T$ and $Q = T$. Therefore, $P \rightarrow Q = T$. So $(P \wedge Q) \Rightarrow (P \rightarrow Q)$ holds.

Problem 4

Assuming that $P_i \rightarrow Q_i(i = 1, ..., n)$, $P_1 \vee P_2 \vee ... \vee P_n$ and $\neg(Q_i \wedge Q_j)(i \neq j)$ are true, prove that $Q_i \rightarrow P_i(i = 1, ..., n)$ are true by deduction in 2.9.

Solution.
We want to prove

$$P_i \rightarrow Q_i|_{i=1,...,n}, P_1 \vee P_2 \vee ... \vee P_n, \neg(Q_i \wedge Q_j)|_{i \neq j} \Rightarrow Q_i \rightarrow P_i|_{i=1,...,n}.$$

$(1)\neg(Q_i \wedge Q_j)|_{i\neq j}$ 前提引入

$(2)Q_i \to \neg Q_j|_{i\neq j}$ (1)置换

$(3)P_j \to Q_j|_{i\neq j}$ 前提引入

$(4)\neg Q_j \to \neg P_j|_{i\neq j}$ (3)置换

$(5)Q_i \to \neg P_j|_{i\neq j}$ (2)(4)三段论

$(6)Q_i$ 附加前提引入

$(7)\neg P_j|_{i\neq j}$ (5)(6)分离

$(8)\neg P_1 \wedge ... \wedge \neg P_{i-1} \wedge \neg P_{i+1} \wedge ... \wedge \neg P_n$ (7)

$(9)\neg(P_1 \vee ... \vee P_{i-1} \vee P_{i+1} \vee ... \vee P_n)$ (8)置换

$(10)P_1 \vee P_2 \vee ... \vee P_n$ 前提引入

$(11)\neg(P_1 \vee ... \vee P_{i-1} \vee P_{i+1} \vee ... \vee P_n) \to P_i$ (10)置换

$(12)P_i$ (9)(11)分离

$(13)Q_i \to P_i$ 条件证明规则

Problem 5

Prove the following theorem by resolution in 2.10.

$$(P \vee Q) \wedge (P \to R) \wedge (Q \to R) \Rightarrow R$$

Solution.
We want to prove $(P \vee Q) \wedge (P \to R) \wedge (Q \to R) \wedge \neg R$ is a contradiction. Firstly, we convert the formula to CNF.

$$\begin{aligned} &(P \vee Q) \wedge (P \to R) \wedge (Q \to R) \wedge \neg R \\ =&(P \vee Q) \wedge (\neg P \vee R) \wedge (\neg Q \vee R) \wedge \neg R \end{aligned}$$

Then we can construct the set of clauses.
S={ $P \vee Q, \neg P \vee R, \neg Q \vee R, \neg R$ }

$(1) P \vee Q$	
$(2) \neg P \vee R$	
$(3) \neg Q \vee R$	
$(4) \neg R$	
$(5) Q \vee R$	(1)(2)归结
$(6) R$	(3)(5)归结
$(7) \square$	(4)(6)归结

Problem 6

Prove

1. $A \to B$ is a tautology iff $B^* \to A^*$ is a tautology
2. $A \to B$ is satisfiable iff $B^* \to A^*$ is satisfiable
3. $A \leftrightarrow B$ is a tautology iff $B^* \leftrightarrow A^*$ is a tautology
4. $A \leftrightarrow B$ is satisfiable iff $B^* \leftrightarrow A^*$ is satisfiable

Solution.

1. If $A \to B$ is a tautology, then $\neg B \to \neg A$ is a tautology. Because $\neg A = A^{*-}$ and $\neg B = B^{*-}$, $B^{*-} \to A^{*-}$ is a tautology. Therefore, $B^* \to A^*$ is a tautology.

   If $B^* \to A^*$ is a tautology, then $(A^*)^* \to (B^*)^*$ is a tautology. Because $A = (A^*)^*$ and $B = (B^*)^*$, $A \to B$ is a tautology.

   In conclusion, $A \to B$ is a tautology iff $B^* \to A^*$ is a tautology.

2. If $A \to B$ is satisfiable, then $\neg B \to \neg A$ is satisfiable. Because $\neg A = A^{*-}$ and $\neg B = B^{*-}$, $B^{*-} \to A^{*-}$ is satisfiable. Therefore, $B^* \to A^*$ is satisfiable.

   If $B^* \to A^*$ is satisfiable, then $B^{*-} \to A^{*-}$ is satisfiable. Because $\neg A = A^{*-}$ and $\neg B = B^{*-}$, $\neg B \to \neg A$ is satisfiable. So $A \to B$ is satisfiable.

   In conclusion, $A \to B$ is satisfiable iff $B^* \to A^*$ is satisfiable.

3. $A \leftrightarrow B$ is a tautology iff $(A \to B) \wedge (B \to A)$ is a tautology. $(A \to B) \wedge (B \to A)$ is a tautology iff $A \to B$ is a tautology and $B \to A$ is a tautology. $A \to B$ and $B \to A$

are tautologies iff $B^* \to A^*$ and $A^* \to B^*$ are tautologies. $B^* \to A^*$ and $A^* \to B^*$ are tautologies iff $B^* \leftrightarrow A^*$ is a tautology.

4. $A \leftrightarrow B$ is satisfiable iff $\neg A \leftrightarrow \neg B$ is satisfiable. Because $\neg A = A^{*-}$ and $\neg B = B^{*-}$, $\neg A \leftrightarrow \neg B$ is satisfiable iff $A^{*-} \leftrightarrow B^{*-}$ is satisfiable. $A^{*-} \leftrightarrow B^{*-}$ is satisfiable iff $B^* \leftrightarrow A^*$ is satisfiable.