# Solution 4

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## Problem 1

Determine if the following formulas are wffs.

1. $(\forall x)(P(x) \wedge R(x)) \to ((\forall x)P(x) \wedge Q(x))$
2. $(\forall x)(P(x) \leftrightarrow Q(x)) \wedge (\exists x)R(x) \wedge S(x)$
3. $(\exists x)((\forall y)P(y) \to Q(x, y))$
4. $(\exists x)(\exists y)(P(x, y, z) \to S(u, v))$
5. $(\forall x)P(x, y) \wedge Q(z)$

**Solution.**

1. No.
2. No.
3. No.

4. Yes.

5. Yes.

# Problem 2

In the following formula, what are bound variables and what are free variables? what is the scope of every quantifier?

$$(\forall x)(P(x) \land Q(x)) \to ((\forall x)R(x) \land Q(z))$$

**Solution.**
Bound variable is "x". Free variable is "z". In $(\forall x)(P(x) \land Q(x))$, $P(x) \land Q(x)$ is the scope of $\forall x$. In $(\forall x)R(x) \land Q(z)$, $R(x)$ is the scope of $\forall x$.

# Problem 3

Represent the following statements with predicate logic formulas. The default domain of discourse includes everything.

1. For every two different points on a plane, there is and only one straight line can pass through them.

2. People working in Shanghai may not live in Shanghai.

3. If it will be fine tomorrow, some students will go swimming.

**Solution.**

1. P(x) denotes x is a point on a plane. Q(x, y, u) denotes u is a straight line passing through two points x and y. EP(x, y) denotes x and y are the same point. EQ(u, v) denotes u and v are the same straight line. So the statement can be represented by

   $$(\forall x)(\forall y)(P(x) \land P(y) \land \neg EP(x,y) \to (\exists u)(Q(x,y,u) \land (\forall v)(Q(x,y,v) \to EQ(u,v)))).$$

2. P(x) denotes x works in Shanghai. Q(x) denotes x lives in Shanghai. The statement can be represented by

   $$(\exists x)(P(x) \land \neg Q(x)).$$

3. r denotes it will be fine tomorrow. P(x) denotes x is a student. Q(x) denotes x will go swimming. The statement can be represented by

   $$r \to (\exists x)(P(x) \land Q(x)).$$

## Problem 4

I(x) means x is a positive integer. P(x) means x is a rational number. What's the meaning of $(\forall x)(I(x) \to P(x)) \wedge \neg(\forall x)(P(x) \to I(x))$.

**Solution.**
All positive integers are rational numbers and not all rational numbers are positive integers.

## Problem 5

1. Write a predicate logic formula. If the domain of discourse is {1,2}, the formula is satisfiable. If the domain of discourse is {1}, the formula is unsatisfiable.
2. If the domain of discourse is {a,b,c}, rewrite $(\exists x)(\forall y)P(x,y)$ and $(\forall x)P(x) \to (\exists y)Q(y)$ like that in 4.5.2 of the textbook.
3. The domain is {a,b}. P(a,a)=T, P(a,b)=F, P(b,a)=F and P(b,b)=T. Determine the truth value of the following formulas:

$$\begin{aligned}
&(\forall x)(\exists y)P(x,y)\\
&(\exists x)(\forall y)P(x,y)\\
&(\forall x)(\forall y)P(x,y)\\
&(\exists x)(\exists y)P(x,y)\\
&(\exists y)\neg P(a,y)\\
&(\exists y)(\forall x)P(x,y)
\end{aligned}$$

**Solution.**

1. $(\exists x)P(x)$. $P(x)$ means $x > 1$.
2. 

$$\begin{aligned}
&(\exists x)(\forall y)P(x,y)\\
=&(P(a,a) \wedge P(a,b) \wedge P(a,c)) \vee\\
&(P(b,a) \wedge P(b,b) \wedge P(b,c)) \vee\\
&(P(c,a) \wedge P(c,b) \wedge P(c,c))
\end{aligned}$$

$$\begin{aligned}
&(\forall x)P(x) \to (\exists y)Q(y)\\
=&(P(a) \wedge P(b) \wedge P(c)) \to (Q(a) \vee Q(b) \vee Q(c))
\end{aligned}$$

3. T, F, F, T, T, F.

## Problem 6

1. Convert the following formulas to PNF according to 5.3.1 in the textbook.

$$(\forall x)(P(x) \to (\exists y)Q(x,y))$$
$$(\exists x)P(x,y) \leftrightarrow (\forall z)Q(z)$$

2. Convert the following formulas to Skolem normal form according to 5.3.2 in the textbook.

$$(\forall x)(P(x) \to (\exists y)Q(x,y)) \lor (\forall z)R(z)$$
$$(\exists y)(\forall x)(\forall z)(\exists u)(\forall v)P(x,y,z,u,v)$$

**Solution.**

1.

$$\begin{aligned}
&(\forall x)(P(x) \to (\exists y)Q(x,y)) \\
=&(\forall x)(\neg P(x) \lor (\exists y)Q(x,y)) \\
=&(\forall x)(\exists y)(\neg P(x) \lor Q(x,y))
\end{aligned}$$

$$\begin{aligned}
&(\exists x)P(x,y) \leftrightarrow (\forall z)Q(z) \\
=&((\exists x)P(x,y) \land (\forall z)Q(z)) \lor (\neg(\exists x)P(x,y) \land \neg(\forall z)Q(z)) \\
=&((\forall x)(\neg P(x,y)) \lor (\forall z)Q(z)) \land ((\exists x)P(x,y) \lor (\exists z)(\neg Q(z))) \\
=&((\forall x)(\neg P(x,y)) \lor (\forall z)Q(z)) \land ((\exists u)P(u,y) \lor (\exists v)(\neg Q(v))) \\
=&(\forall x)(\forall z)(\exists u)(\exists v)((\neg P(x,y) \lor Q(z)) \land (P(u,y) \lor (\neg Q(v))))
\end{aligned}$$

2.

$$\begin{aligned}
&(\forall x)(P(x) \to (\exists y)Q(x,y)) \lor (\forall z)R(z) \\
=&(\forall x)(\neg P(x) \lor (\exists y)Q(x,y)) \lor (\forall z)R(z) \\
=&(\forall x)(\exists y)(\forall z)(\neg P(x) \lor Q(x,y) \lor R(z)) \\
Skolem\ form\ :&(\forall x)(\forall z)(\neg P(x) \lor Q(x,f(x)) \lor R(z))
\end{aligned}$$

$$\begin{aligned}
&(\exists y)(\forall x)(\forall z)(\exists u)(\forall v)P(x,y,z,u,v) \\
Skolem\ form\ :&(\forall x)(\forall z)(\forall v)P(x,a,z,f(x,z),v)
\end{aligned}$$

## Problem 7

If we use GSAT to solve $(P \vee \neg Q) \wedge (\neg P \vee Q) \wedge (P \vee \neg Q) \wedge \neg P \wedge \neg Q$, give all the starting points that make GSAT return unknown. In this problem, when will GSAT return unsat? When will GSAT return sat?

**Solution.**
The profit under different assignments are shown below.

| | Q=T | Q=F |
|---|---|---|
| P=T | 3 | 3 |
| P=F | 2 | 5 |

Assuming $m > 1$ in GSAT algorithm. According to the table, GSAT will not return unknown. Because GSAT is not a complete algorithm, it never return unsat. GSAT will return sat if the starting point is TT, TF, FT and FF.

## Problem 8

Use DPLL to solve this problem. When splitting on decision literals, follow this initial order: 1 false.

$$\emptyset \parallel \; 1 \vee \overline{2},\; \overline{1} \vee \overline{2},\; 2 \vee 3,\; \overline{3} \vee 4,\; 1 \vee \overline{4}$$

**Solution.**

$$\begin{aligned}
\emptyset \parallel &\; 1 \vee \overline{2},\; \overline{1} \vee \overline{2},\; 2 \vee 3,\; \overline{3} \vee 4,\; 1 \vee \overline{4} \Rightarrow (Decide)\\
\overline{1}^d \parallel &\; 1 \vee \overline{2},\; \overline{1} \vee \overline{2},\; 2 \vee 3,\; \overline{3} \vee 4,\; 1 \vee \overline{4} \Rightarrow (UnitPropagate)\\
\overline{1}^d\, \overline{2} \parallel &\; 1 \vee \overline{2},\; \overline{1} \vee \overline{2},\; 2 \vee 3,\; \overline{3} \vee 4,\; 1 \vee \overline{4} \Rightarrow (UnitPropagate)\\
\overline{1}^d\, \overline{2}\, \overline{4} \parallel &\; 1 \vee \overline{2},\; \overline{1} \vee \overline{2},\; 2 \vee 3,\; \overline{3} \vee 4,\; 1 \vee \overline{4} \Rightarrow (UnitPropagate)\\
\overline{1}^d\, \overline{2}\; \overline{4}\, 3 \parallel &\; 1 \vee \overline{2},\; \overline{1} \vee \overline{2},\; 2 \vee 3,\; \overline{3} \vee 4,\; 1 \vee \overline{4} \Rightarrow (Backtrack)\\
1 \parallel &\; 1 \vee \overline{2},\; \overline{1} \vee \overline{2},\; 2 \vee 3,\; \overline{3} \vee 4,\; 1 \vee \overline{4} \Rightarrow (UnitPropagate)\\
1\, \overline{2} \parallel &\; 1 \vee \overline{2},\; \overline{1} \vee \overline{2},\; 2 \vee 3,\; \overline{3} \vee 4,\; 1 \vee \overline{4} \Rightarrow (UnitPropagate)\\
1\, \overline{2}\, 3 \parallel &\; 1 \vee \overline{2},\; \overline{1} \vee \overline{2},\; 2 \vee 3,\; \overline{3} \vee 4,\; 1 \vee \overline{4} \Rightarrow (UnitPropagate)\\
1\, \overline{2}\, 3\, 4 \parallel &\; 1 \vee \overline{2},\; \overline{1} \vee \overline{2},\; 2 \vee 3,\; \overline{3} \vee 4,\; 1 \vee \overline{4}
\end{aligned}$$