Solution 5

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Problem 1

Prove the following equations and tautological implication. (Hint:5.1, 5.2, 5.4.2 in the textbook)

1.

$$\begin{aligned}&\neg(\exists x)(\exists y)(P(x) \wedge P(y) \wedge Q(x) \wedge Q(y) \wedge R(x, y))\\=&(\forall x)(\forall y)((P(x) \wedge P(y) \wedge Q(x) \wedge Q(y)) \rightarrow \neg R(x, y))\end{aligned}$$

2.

$$\begin{aligned}&(\exists x)(P(x) \rightarrow Q(x))\\=&(\forall x)P(x) \rightarrow (\exists x)Q(x)\end{aligned}$$

3.

$$\begin{aligned}&(\forall y)(\exists x)((P(x) \rightarrow q) \vee S(y))\\=&((\forall x)P(x) \rightarrow q) \vee (\forall y)S(y)\end{aligned}$$

4.

$$
\begin{aligned}
&(\exists x)P(x) \to (\forall x)Q(x) \\
\Rightarrow &(\forall x)(P(x) \to Q(x))
\end{aligned}
$$

Solution.

1.

$$
\begin{aligned}
&\neg(\exists x)(\exists y)(P(x) \wedge P(y) \wedge Q(x) \wedge Q(y) \wedge R(x,y)) \\
=&(\forall x)(\forall y)(\neg(P(x) \wedge P(y) \wedge Q(x) \wedge Q(y) \wedge R(x,y))) \\
=&(\forall x)(\forall y)(\neg(P(x) \wedge P(y) \wedge Q(x) \wedge Q(y)) \vee \neg R(x,y)) \\
=&(\forall x)(\forall y)((P(x) \wedge P(y) \wedge Q(x) \wedge Q(y)) \to \neg R(x,y))
\end{aligned}
$$

2.

$$
\begin{aligned}
&(\exists x)(P(x) \to Q(x)) \\
=&(\exists x)(\neg P(x) \vee Q(x)) \\
=&\neg(\forall x)P(x) \vee (\exists x)Q(x) \\
=&(\forall x)P(x) \to (\exists x)Q(x)
\end{aligned}
$$

3.

$$
\begin{aligned}
&(\forall y)(\exists x)((P(x) \to q) \vee S(y)) \\
=&(\exists x)(P(x) \to q) \vee (\forall y)S(y) \\
=&((\forall x)P(x) \to q) \vee (\forall y)S(y)
\end{aligned}
$$

4.

$$
\begin{aligned}
&(\exists x)P(x) \to (\forall x)Q(x) \\
=&\neg(\exists x)P(x) \vee (\forall x)Q(x) \\
=&(\forall x)(\neg P(x)) \vee (\forall x)Q(x) \\
\Rightarrow&(\forall x)((\neg P(x)) \vee Q(x)) \\
=&(\forall x)(P(x) \to Q(x))
\end{aligned}
$$

Problem 2

1. Prove $(\forall x)(P(x) \vee Q(x)) \wedge (\forall x)(Q(x) \rightarrow \neg R(x)) \Rightarrow (\exists x)(R(x) \rightarrow P(x))$ by deduction in 5.5.

2. Every student in the university is either an undergraduate or a postgraduate. Some students are male. John is not a postgraduate but he is male. Therefore, if John is a student in the university, he must be an undergraduate. Represent these statements in predicate logic and prove the conclusion ("if John is a student in the university, he must be an undergraduate") by resolution method in 5.6

Solution.

1.

$$
\begin{array}{lr}
(1)(\forall x)(P(x) \vee Q(x)) & \text{前提} \\
(2)(\forall x)(Q(x) \rightarrow \neg R(x)) & \text{前提} \\
(3)P(x) \vee Q(x) & (1)\text{全称量词消去} \\
(4)Q(x) \rightarrow \neg R(x) & (2)\text{全称量词消去} \\
(5)\neg Q(x) \rightarrow P(x) & (3)\text{置换} \\
(6)R(x) \rightarrow \neg Q(x) & (4)\text{置换} \\
(7)R(x) \rightarrow P(x) & (5)(6)\text{三段论} \\
(8)(\exists x)(R(x) \rightarrow P(x)) & (7)\text{存在量词引入}
\end{array}
$$

2. P(x) denotes x is a student in the university. Q(x) denotes x is an undergraduate. R(x) denotes x is a postgraduate. S(x) denotes x is male. We want to prove that

$$
\begin{aligned}
&(\forall x)(P(x) \rightarrow ((Q(x) \vee R(x)) \wedge (\neg Q(x) \vee \neg R(x)))) \wedge (\exists x)(P(x) \wedge S(x)) \wedge (\neg R(John) \wedge S(John)) \\
&\Rightarrow P(John) \rightarrow Q(John)
\end{aligned}
$$

Then we construct the set of clauses.

$$
\begin{aligned}
&(\forall x)(P(x) \rightarrow ((Q(x) \vee R(x)) \wedge (\neg Q(x) \vee \neg R(x)))) \wedge (\exists x)(P(x) \wedge S(x)) \wedge (\neg R(John) \wedge S(John)) \\
=&(\forall x)(\neg P(x) \vee ((Q(x) \vee R(x)) \wedge (\neg Q(x) \vee \neg R(x)))) \wedge (\exists x)(P(x) \wedge S(x)) \wedge (\neg R(John) \wedge S(John)) \\
=&(\forall x)((\neg P(x) \vee Q(x) \vee R(x)) \wedge (\neg P(x) \vee \neg Q(x) \vee \neg R(x))) \wedge \\
&(\exists x)(P(x) \wedge S(x)) \wedge (\neg R(John) \wedge S(John))
\end{aligned}
$$

So the set of clauses

$$S = \{\neg P(x) \vee Q(x) \vee R(x), \neg P(x) \vee \neg Q(x) \vee \neg R(x), P(a), S(a), \neg R(John), S(John), P(John), \neg Q(John)\}$$

$(1) \neg P(x) \lor Q(x) \lor R(x)$

$(2) \neg P(x) \lor \neg Q(x) \lor \neg R(x)$

$(3) P(a)$

$(4) S(a)$

$(5) \neg R(John)$

$(6) S(John)$

$(7) P(John)$

$(8) \neg Q(John)$

$(9) Q(John) \lor R(John)$ (1)(7)归结

$(10) Q(John)$ (5)(9)归结

$(11) \Box$ (8)(10)归结

Problem 3

Determine if the following deduction are right. Explain your reasons if the deduction is wrong.

1. If $(\forall x)P(x) \to Q(x)$, then $P(a) \to Q(a)$.
2. If $\neg(\exists x)(\neg P(x) \land \neg Q(x))$, then $\neg((\exists x)\neg P(x) \land (\exists x)\neg Q(x))$.
3. If $(\exists x)(\neg P(x) \land \neg Q(x))$, then $(\exists x)\neg P(x) \land (\exists x)\neg Q(x)$.

Solution.

1. False. $(\forall x)P(x) \to Q(x)$ is not a wff.
2. False. $\neg(\exists x)(\neg P(x) \land \neg Q(x)) = (\forall x)(P(x) \lor Q(x))$, $\neg((\exists x)\neg P(x) \land (\exists x)\neg Q(x)) = (\forall x)P(x) \lor (\forall x)Q(x)$. $(\forall x)(P(x) \lor Q(x)) \Rightarrow (\forall x)P(x) \lor (\forall x)Q(x)$ is not correct.
3. True.

Problem 4

Determine if the following formulas are universally valid. If they are universally valid, give the proof. Otherwise, give the counterexample.

1. $((\exists x)P(x) \to (\exists x)Q(x)) \to (\exists x)(P(x) \to Q(x))$
2. $(\forall x)(\exists y)P(x, y) \to (\exists y)(\forall x)P(x, y)$

Solution.

1. It's universally valid.

$$
\begin{aligned}
&((\exists x)P(x) \to (\exists x)Q(x)) \to (\exists x)(P(x) \to Q(x)) \\
=&\neg(\neg(\exists x)P(x) \vee (\exists x)Q(x)) \vee (\exists x)(\neg P(x) \vee Q(x)) \\
=&((\exists x)P(x) \wedge \neg(\exists x)Q(x)) \vee (\exists x)(\neg P(x)) \vee (\exists x)Q(x) \\
=&((\exists x)P(x) \vee (\exists x)(\neg P(x))) \wedge (\neg(\exists x)Q(x) \vee (\exists x)(\neg P(x))) \vee (\exists x)Q(x) \\
=&\neg(\exists x)Q(x) \vee (\exists x)(\neg P(x)) \vee (\exists x)Q(x) \\
=&T
\end{aligned}
$$

2. It is not universally valid. Assuming the domain of discourse is $\{1, 2\}$, P(1,2)=P(2,1)=F and P(1,1)=P(2,2)=T. Then $(\forall x)(\exists y)P(x, y)$ is T and $(\exists y)(\forall x)P(x, y)$ is F. The formula is not valid.

Problem 5

Encode $f(b) = b, f(b) = f(a), a = c, f(c) \neq b$ into a SAT problem and determine the satisfiability of it. You don't need to write the process of DPLL. Directly write its satisfiability.

Solution.
We can use f_a, f_b and f_c to denote $f(a), f(b)$ and $f(c)$. Then the formula can be encoded into

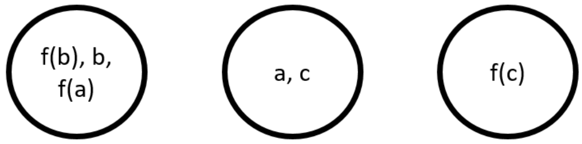
$$
\begin{aligned}
&(f_b \leftrightarrow b)\wedge \\
&(f_b \leftrightarrow f_a)\wedge \\
&(a \leftrightarrow c)\wedge \\
&\neg(f_c \leftrightarrow b)\wedge \\
&((a \leftrightarrow b) \to (f_a \leftrightarrow f_b))\wedge \\
&((a \leftrightarrow c) \to (f_a \leftrightarrow f_c))\wedge \\
&((b \leftrightarrow c) \to (f_b \leftrightarrow f_c))
\end{aligned}
$$

The formula is unsatisfiable.

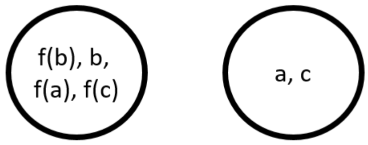
Problem 6

Describe how to solve $f(b) = b, f(b) = f(a), a = c, b \neq c, f(c) \neq a$ by EUF solver. Draw the graph like that in slides.

Solution.



Because $(a = c) \rightarrow (f(a) = f(c))$,



EUF solver returns sat.

Problem 7

Solve $f(i) - f(j) \neq 0 \wedge i - j = 0$ by the Nelson-Oppen Method.

Solution.
The formula is converted into

$$\neg(m - n = 0) \wedge i - j = 0 \wedge m = f(i) \wedge n = f(j).$$

The shared constants S ={ m, n, i, j}.$\neg(m - n = 0) \wedge i - j = 0$ is assigned to arithmetic solver and $m = f(i) \wedge n = f(j)$ is assigned to EUF solver. We need to consider all possible equivalence relations on S. There are 15 possible relations.

So the formula is unsatisfiable.

m=n	m=i	m=j	n=i	n=j	i=j	arithmetic solver	EUF solver
T	T	T	T	T	T	unsat	sat
T	F	F	F	F	T	unsat	sat
F	T	F	F	T	F	unsat	sat
F	F	T	T	F	F	unsat	sat
T	T	F	T	F	F	unsat	sat
T	F	T	F	T	F	unsat	sat
F	F	F	T	T	T	sat	unsat
F	T	T	F	F	T	sat	unsat
F	F	F	F	F	F	unsat	sat
T	F	F	F	F	F	unsat	sat
F	T	F	F	F	F	unsat	sat
F	F	T	F	F	F	unsat	sat
F	F	F	T	F	F	unsat	sat
F	F	F	F	T	F	unsat	sat
F	F	F	F	F	T	sat	unsat