Binary Relation

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Adapted From:

Stanford CS103 Binary Relation

《数理逻辑与集合论》9.3.4,10.1-10.4,10.5.1,10.6,10.7

Review

- **ZF Axiom System**
  - Russell's paradox in naïve set theory.
  - Subset axiom schema, Axiom of regularity
- **Cardinality of Sets**
  - Cardinality of finite sets. Principle of inclusion and exclusion.
  - Cardinality of infinite sets. Comparing cardinality.
  - Cantor's Theorem.
  - $|Problems| > |Programs|$

How we investigate computability?

2.1 Discussion on Problems

Part1. Intro & Set theory⟨I⟩ : Basics & Formal Language.

Part2. Set Theory⟨II⟩: Axiom system & Cardinality.

Part3. Capture Structures: Binary Relation & Function

2.2 Discussion on Computation

Part1. Turing Machine Basics.

Part2. Variants of Turing Machine. Church-Turing Thesis.

2.3 Discussion on computability

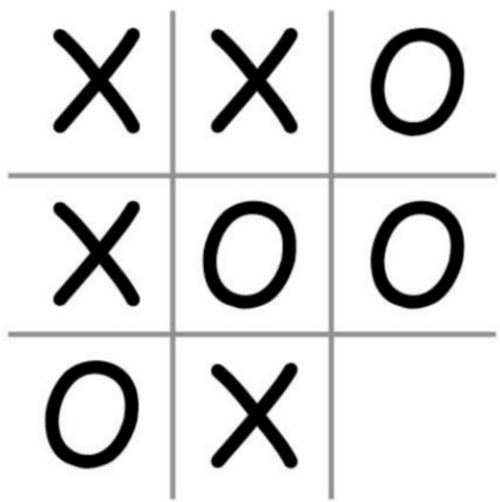
Part1. The Language of Turing Machine. R & RE

Part2. Undecidability.

What is the structure of a problem?

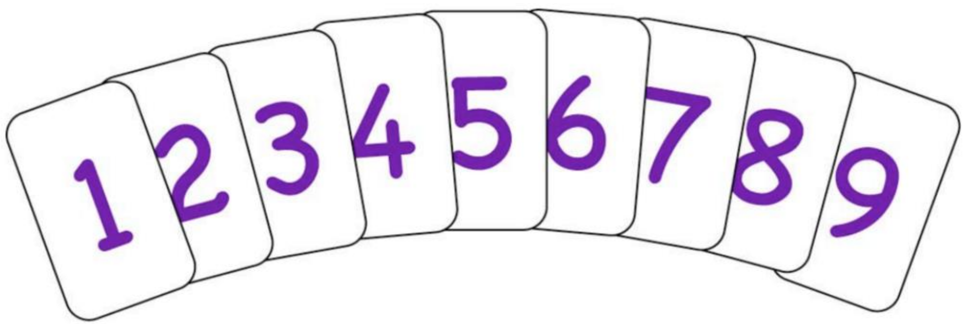
Tic Tac Toe

- Two players take turns marking the spaces in a 3×3 grid.
- The player who succeeds in placing three of their marks in a horizontal, vertical, or diagonal row is the winner.



15 Points

- Two players take turns to select a card with points from 1 to 9.
- The player who picks three cards whose sum equals 15 wins.



Take a deeper look.

X	X	O
X	O	O
O	X	

8	1	6
3	5	7
4	9	2

They only differ in their surface structure, but are based on an identical deep structure!!

Structure of a Problems

- A collection of objects.
  - Cells in Tic Tac Toe.
  - Cards in 15 points.
  - Students in cdm class.
- The connection between objects.
  - Three cells are in a same horizontal, vertical, or diagonal row.
  - Sum of three cards equals 15.
  - Two students come from the same city.

Structure of a Problems

- A collection of objects.
  - Cells in Tic Tac Toe.
  - Cards in 15 points.
  - Students in cdm class.

Set Theory

- The connection between objects.
  - Three cells are in a same horizontal, vertical, or diagonal row.
  - Sum of three cards equals 15.
  - Two students are in same desk.

Structure of a Problems

- A collection of objects.
  - Cells in Tic Tac Toe.
  - Cards in 15 points.
  - Students in cdm class.
- The connection between objects.
  - Three cells are in a same horizontal, vertical, or diagonal row.
  - Sum of three cards equals 15.
  - Two students are in same desk.

Relation & Function

What are relations?

Relations

- We've seen a lot of relations:
  - Between sets:

$$A = B, A \subseteq B, A \neq B$$

  - Between numbers:

$x \equiv_k y$ means x%k=y%k

$$x < y, x = y, x \equiv_k y$$

  - Between cards:

$$c_1 + c_2 + c_3 = 15$$

Relations

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  - Between numbers:

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  - Between cards:

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Connection between two objects
Binary Relation

Relations

- We've seen a lot of relations:
  - Between sets:
    $$A = B, A \subseteq B, A \neq B$$
  - Between numbers:
    $$x < y, x = y, x \equiv_k y$$
  - Between cards:
    $$c_1 + c_2 + c_3 = 15$$

Connection between three objects
Trinary Relation

Relations

- We’ve seen a lot of relations:
  - Between sets:
$$A = B, A \subseteq B, A \neq B$$
  - Between numbers:
$$x < y, x = y, x \equiv_k y$$
  - Between cards:
$$c_1 + c_2 + c_3 = 15$$
- **We only focus on binary relations.**

Ordered Pair

Ordered pair $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$ is collection of two elements $\boldsymbol{a}, \boldsymbol{b}$ that has $\boldsymbol{a}$ as its first element and $\boldsymbol{b}$ as its second elements.

- Ordered pair $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$ satisfies:
  - $\boldsymbol{a} \neq \boldsymbol{b} \Rightarrow \langle \boldsymbol{a}, \boldsymbol{b} \rangle \neq \langle \boldsymbol{b}, \boldsymbol{a} \rangle$
  - $\langle \boldsymbol{a}, \boldsymbol{b} \rangle = \langle \boldsymbol{c}, \boldsymbol{d} \rangle \Leftrightarrow \boldsymbol{a} = \boldsymbol{c} \wedge \boldsymbol{b} = \boldsymbol{d}$
- Examples:
  - $\langle 1,2 \rangle$ and $\langle 2,1 \rangle$ are two different ordered pair.
  - $\langle 1,1 \rangle$ is also an ordered pair.

Ordered Pair and Set

- We can use set to describe an ordered pair.
- Define that : $\langle \boldsymbol{a}, \boldsymbol{b} \rangle = \{\boldsymbol{a}, \{\boldsymbol{a}, \boldsymbol{b}\}\}$
- If $\boldsymbol{a} \neq \boldsymbol{b}$:

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle = \{\boldsymbol{a}, \{\boldsymbol{a}, \boldsymbol{b}\}\} \neq \{\boldsymbol{b}, \{\boldsymbol{b}, \boldsymbol{a}\}\} = \langle \boldsymbol{b}, \boldsymbol{a} \rangle$$

- If $\langle \boldsymbol{a}, \boldsymbol{b} \rangle = \langle \boldsymbol{c}, \boldsymbol{d} \rangle$:

$$\{\boldsymbol{a}, \{\boldsymbol{a}, \boldsymbol{b}\}\} = \{\boldsymbol{c}, \{\boldsymbol{c}, \boldsymbol{d}\}\} \Leftrightarrow \boldsymbol{a} = \boldsymbol{c} \wedge \boldsymbol{b} = \boldsymbol{d}$$

Cartesian Products

The Cartesian product of the sets $\boldsymbol{A},\boldsymbol{B}$, donated by $\boldsymbol{A} \times \boldsymbol{B}$, is the set of ordered pair $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$ where $\boldsymbol{a} \in \boldsymbol{A}$ and $\boldsymbol{b} \in \boldsymbol{B}$

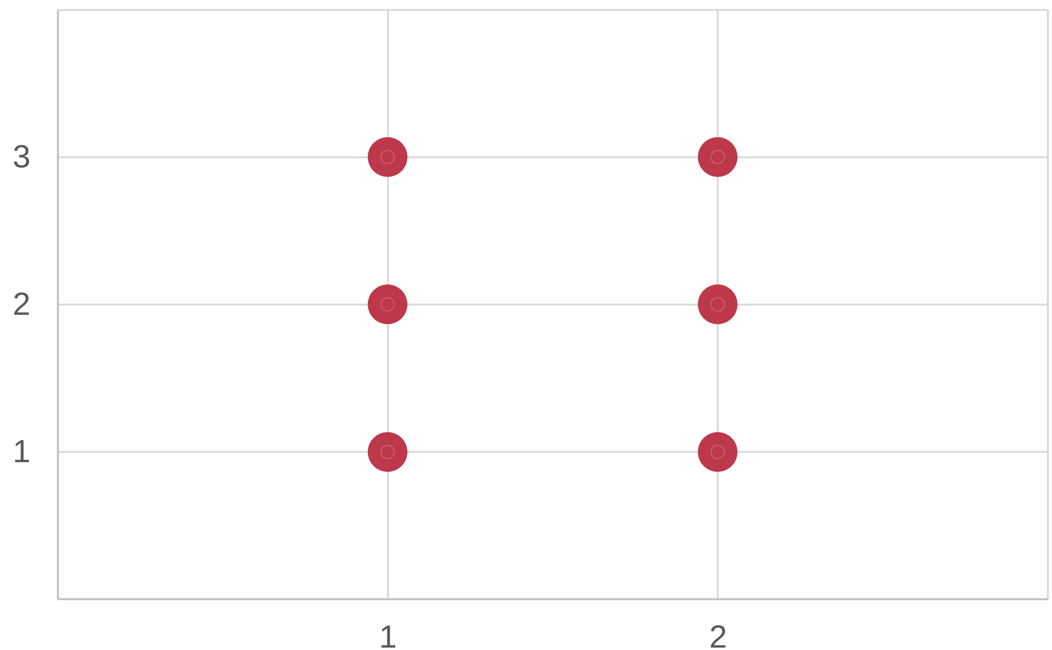
$$\boldsymbol{A} \times \boldsymbol{B} = \{\langle \boldsymbol{a}, \boldsymbol{b} \rangle | \boldsymbol{a} \in \boldsymbol{A} \wedge \boldsymbol{b} \in \boldsymbol{B}\}$$

- Let $A = \{1,2\}$,$B = \{1,2,3\}$
  - $\emptyset \times A = \mathrm{B} \times \emptyset = \emptyset$
  - $A \times B = \{ \langle 1,1 \rangle, \langle 1,2 \rangle, \langle 1,3 \rangle, \langle 2,1 \rangle, \langle 2,2 \rangle, \langle 2,3 \rangle\}$
  - $B \times A = \{ \langle 1,1 \rangle, \langle 1,2 \rangle, \langle 2,1 \rangle, \langle 2,2 \rangle, \langle 3,1 \rangle, \langle 3,2 \rangle\} \neq A \times B$

Cartesian Products

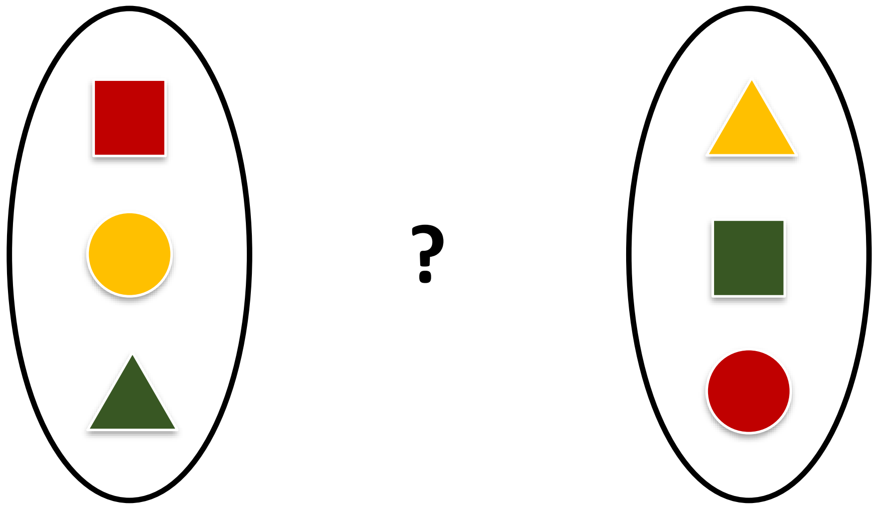
$A \times B$

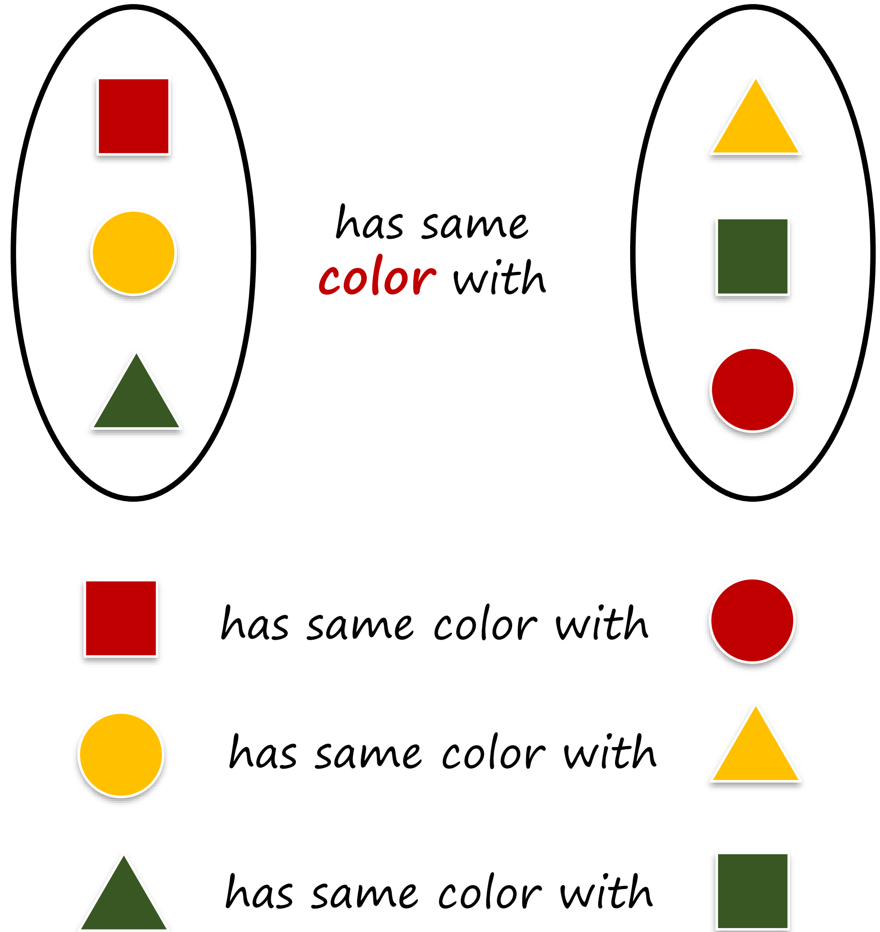
$B = \{1,2,3\}$



$A = \{1,2\}$

What exactly is a binary relation?

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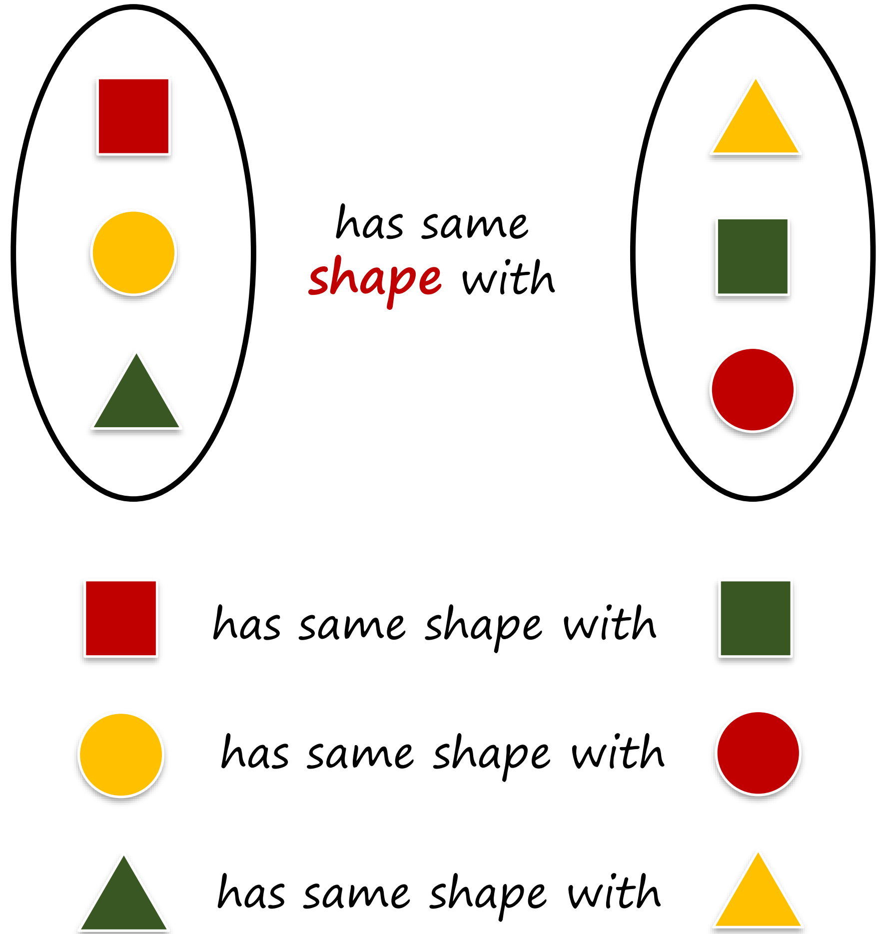
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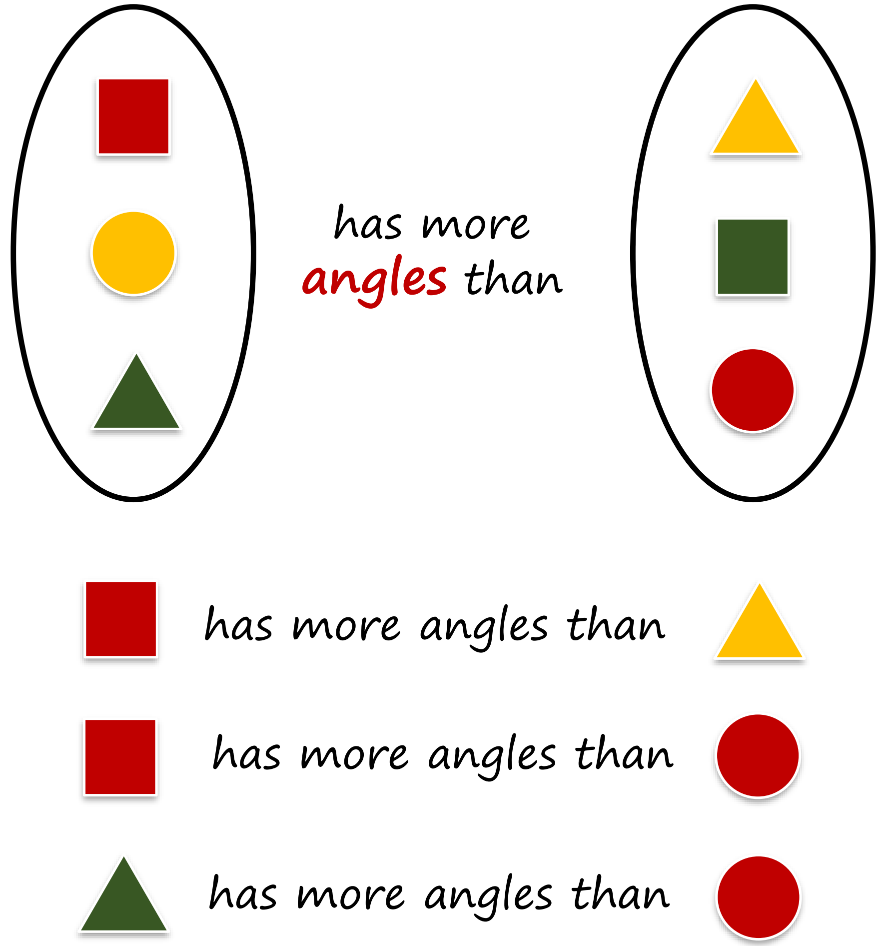
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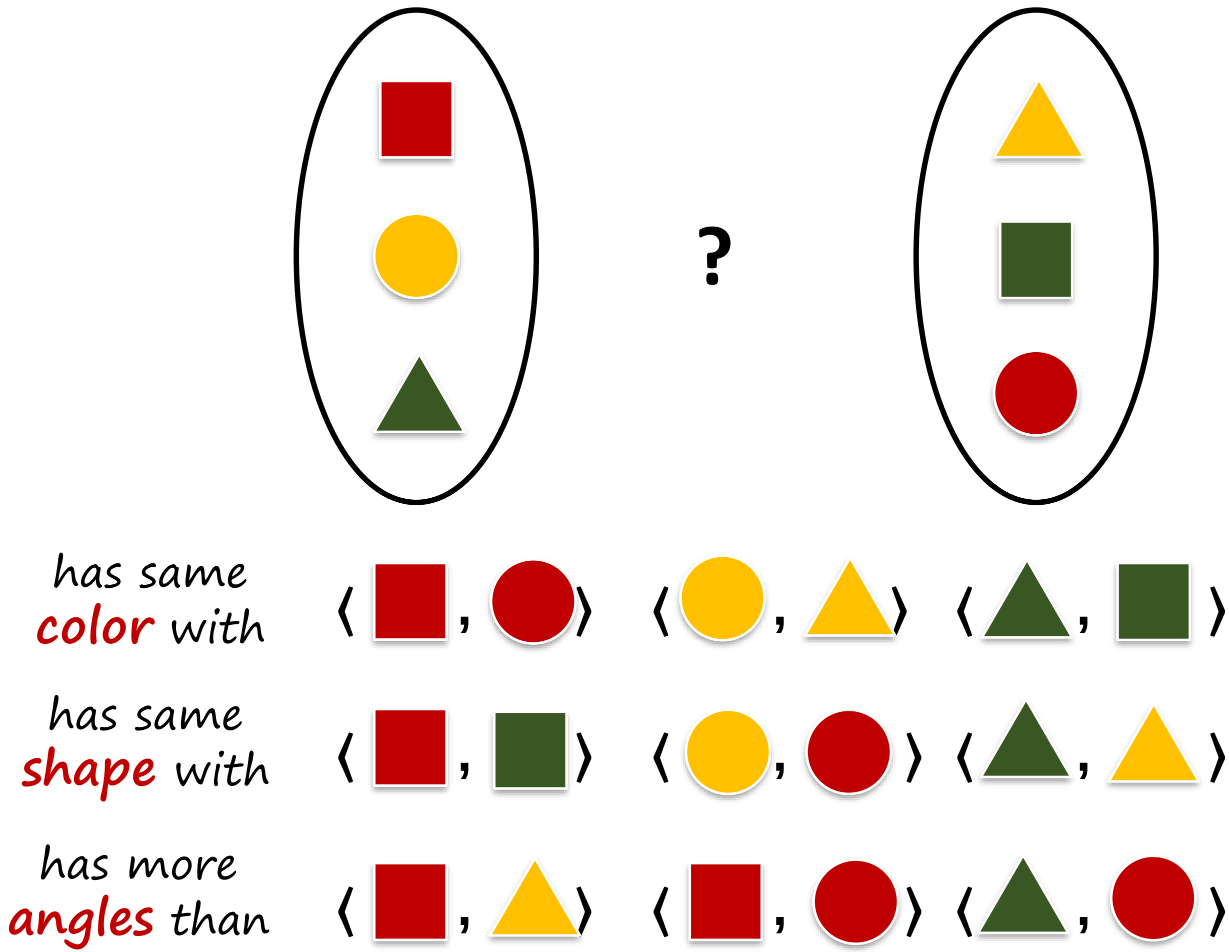
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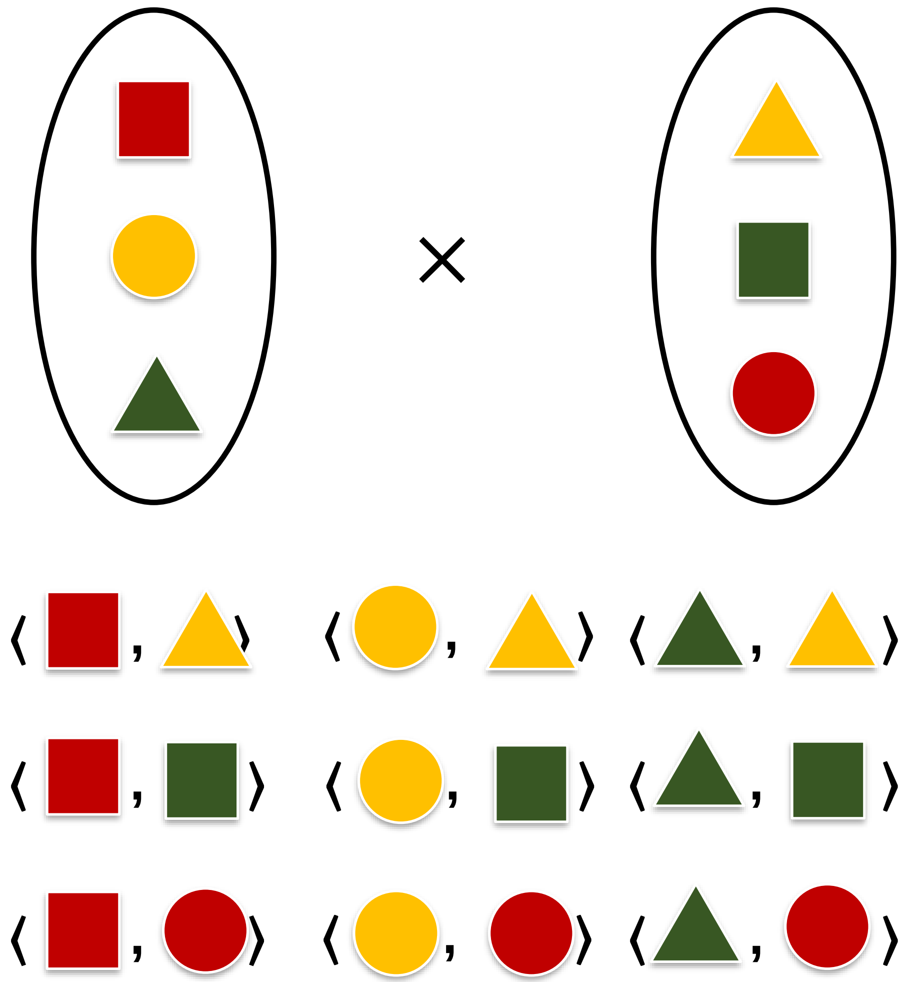

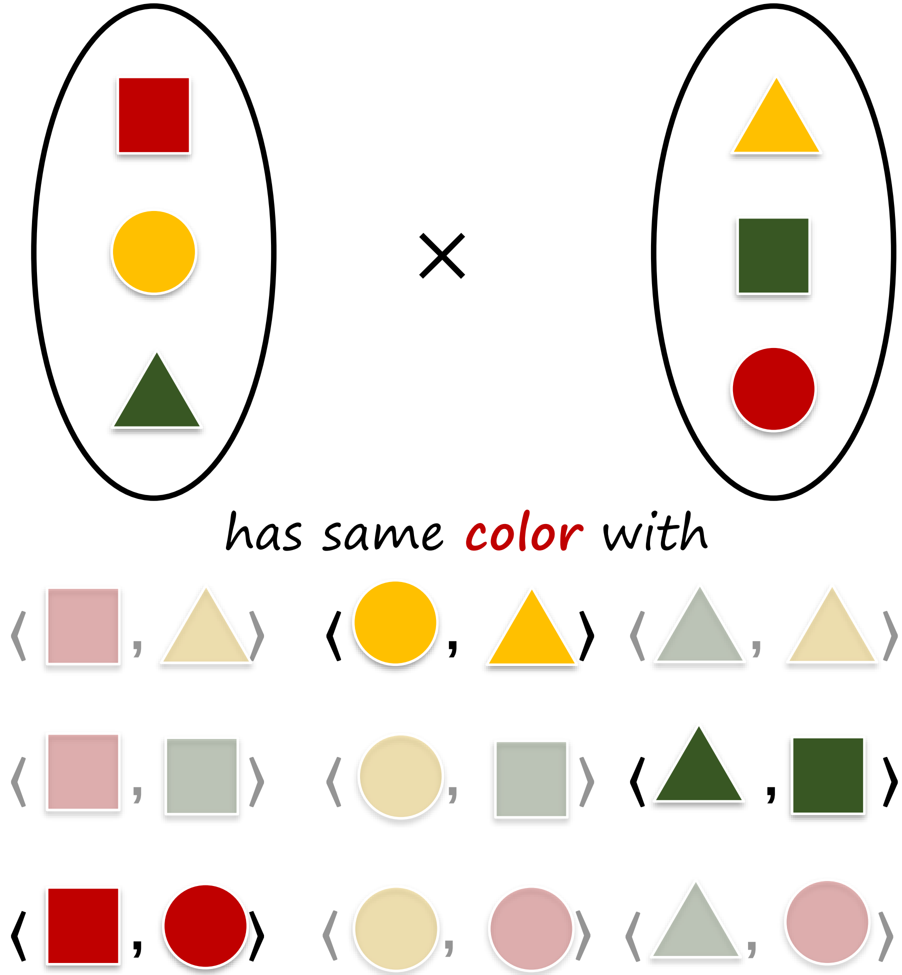
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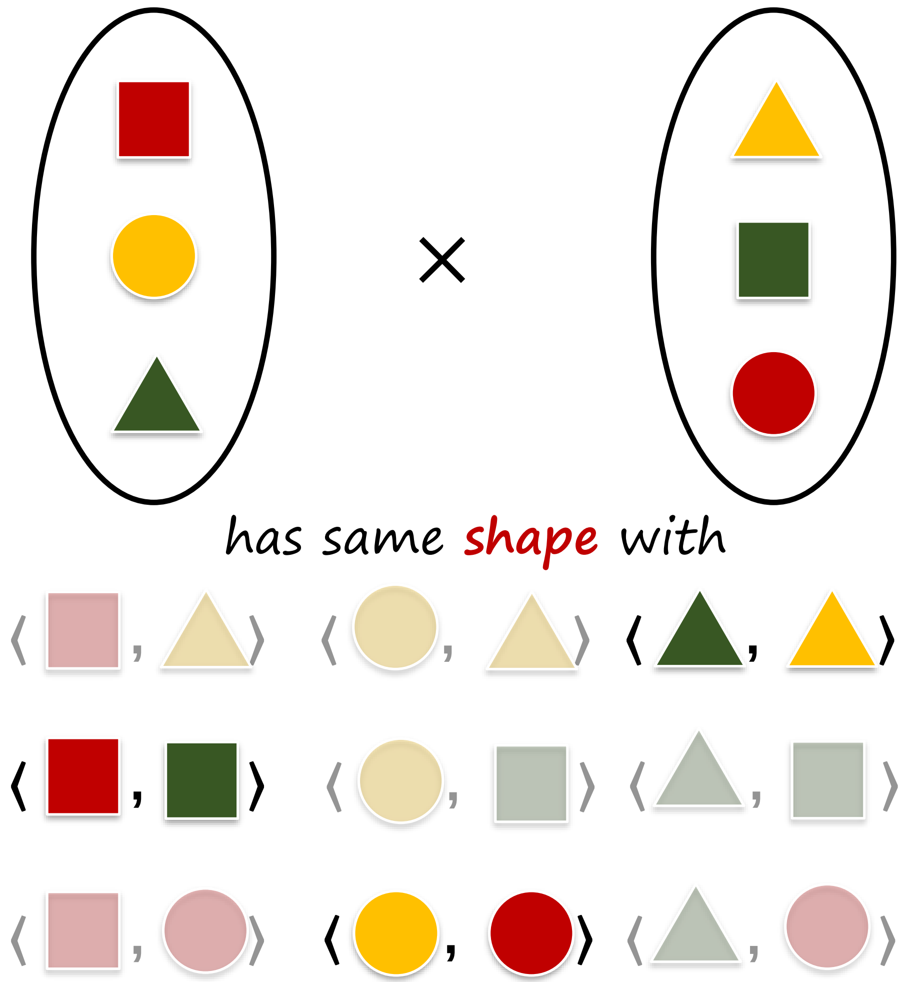
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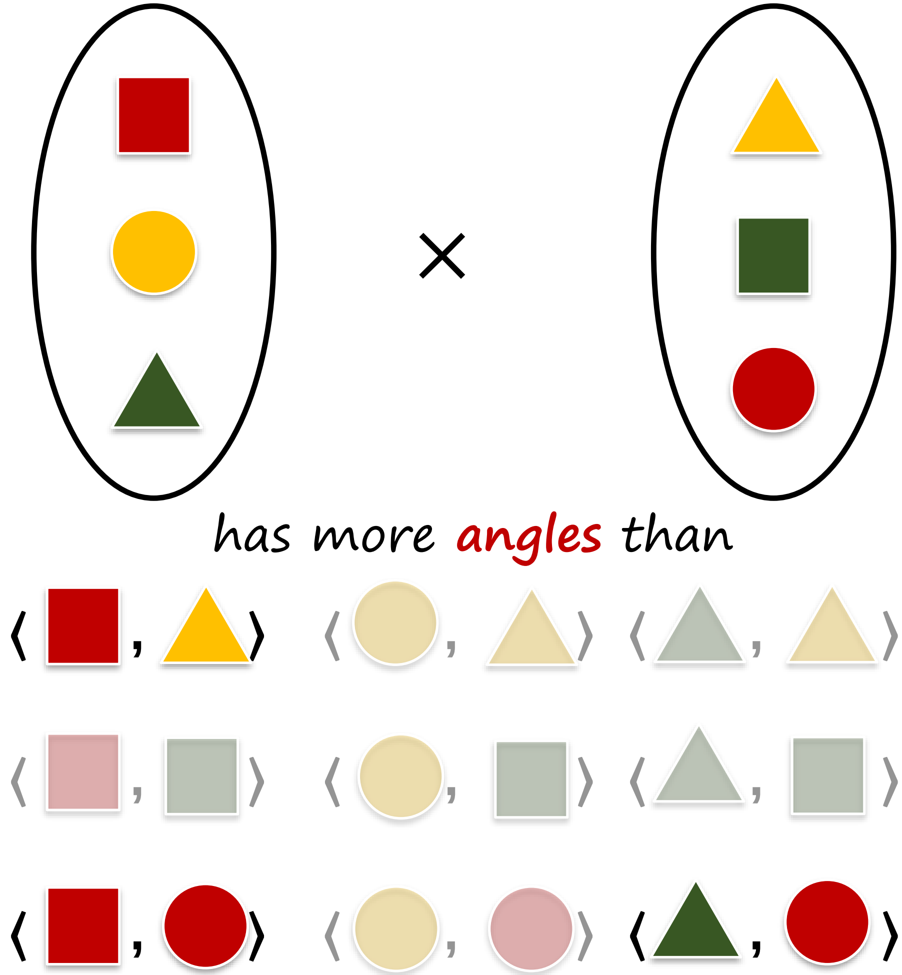
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Binary Relation

Let $\boldsymbol{A}$,$\boldsymbol{B}$ be sets. A binary relation from $\boldsymbol{A}$ to $\boldsymbol{B}$ is a subset of $\boldsymbol{A} \times \boldsymbol{B}$.

- In other words, a binary relation from $\boldsymbol{A}$ to $\boldsymbol{B}$ is a set $\boldsymbol{R}$ of **ordered pairs**, where the first element of each ordered pair comes from $\boldsymbol{A}$ and the second element comes from $\boldsymbol{B}$.
  – $\boldsymbol{aRb}$ means $\langle \boldsymbol{a}, \boldsymbol{b} \rangle \in \boldsymbol{R}$
  – $\boldsymbol{a \not R b}$ means $\langle \boldsymbol{a}, \boldsymbol{b} \rangle \notin \boldsymbol{R}$
- A relation on a set $\boldsymbol{A}$ is a relation from $\boldsymbol{A}$ to $\boldsymbol{A}$, or is a subset of $\boldsymbol{A} \times \boldsymbol{A}$.

Domain, Range and Field

- For a relation $\boldsymbol{R}$ from $\boldsymbol{A}$ to $\boldsymbol{B}$, we define:
  - The domain(定义域)of R is:
$$dom(R) = \{x \mid (\exists y)(\langle x, y\rangle \in R)\}$$
  - The range(值域)of R is:
$$ran(R) = \{y \mid (\exists x)(\langle x, y\rangle \in R)\}$$
  - The field(域)of R is:
$$fld(R) = dom(R) \cup ran(R)$$

Binary Relation

- For set $\boldsymbol{A} = \{1,2,3\}$, define:
  - $R_1 = \{\langle x, y\rangle | x \in A \wedge y \in A \wedge x|y\}$
  - $R_2 = \{\langle x, y\rangle | x \in A \wedge y \in A \wedge x < y\}$
- Then
  - $1R_1 2$, $2R_2 3$
  - $R_1 = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 2,2\rangle, \langle 3,3\rangle\}$
  - $R_2 = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 2,3\rangle\}$
  - $dom(R_1) = \{1,2,3\}, ran(R_1) = \{1,2,3\}, fld(R_1) = \{1,2,3\}$
  - $dom(R_2) = \{1,2\}, ran(R_2) = \{2,3\}, fld(R_2) = \{1,2,3\}$

Special Relations

- For any set $\boldsymbol{A}$, there are three kinds of special relations.
  - The identity relation （恒等关系）：
$$\boldsymbol{I_A} = \{\langle \boldsymbol{x}, \boldsymbol{x}\rangle | \boldsymbol{x} \in \boldsymbol{A}\}$$
  - The universal relation （全域关系）：
$$\boldsymbol{E_A} = \{\langle \boldsymbol{x}, \boldsymbol{y}\rangle | \boldsymbol{x} \in \boldsymbol{A} \wedge \boldsymbol{y} \in \boldsymbol{A}\}$$
  - The empty relation （空关系）：
$$\boldsymbol{N_A} = \emptyset$$

- For arbitrary elements $\boldsymbol{x} \in \boldsymbol{A}$ and $\boldsymbol{y} \in \boldsymbol{A}$:
  - xN_Ay holds never
  - xE_Ay holds always
  - xI_Ay holds if and only if $x = y$

Relation Matrix(关系矩阵)

- Suppose that $\boldsymbol{R}$ is a relation from $A = \{a_1, a_2, a_3, \ldots, a_m\}$ to $B = \{b_1, b_2, b_3 \ldots, b_n\}$. Then the relation R can be represented by zero-one matrix $M_R = (m_{ij})_{m\times n}$, where:

$$m_{ij} = \begin{cases} 1 & if \langle a_i, b_j \rangle \in R \\ 0 & if \langle a_i, b_j \rangle \notin R \end{cases}$$

- If $\boldsymbol{R}$ is a relation on $\boldsymbol{A} = \{\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3} \ldots \boldsymbol{a_m}\}$, then the relation matrix of $\boldsymbol{R}$ is a $m \times m$ square $M_R = (m_{ij})_{m\times m}$

Relation Matrix: Example

Let $A = \{1, 2, 3, 4\}$, the relation R_1 on A is:

$$R_1 = \{\langle 1, 1\rangle, \langle 1, 3\rangle, \langle 2, 1\rangle, \langle 2, 3\rangle, \langle 2, 4\rangle, \langle 3, 1\rangle, \langle 3, 2\rangle, \langle 4, 1\rangle\}$$

The relation matrix of R_1 is:

$$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Relation Matrix: Example

Identity relation $\boldsymbol{I}$:

$$M_I = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

Relation Matrix: Example

Universal relation $\boldsymbol{E}$:

$$M_E = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 1 & 1 & \ldots & 1 \end{bmatrix}$$

Relation Matrix: Example

Empty relation $\boldsymbol{N}$:

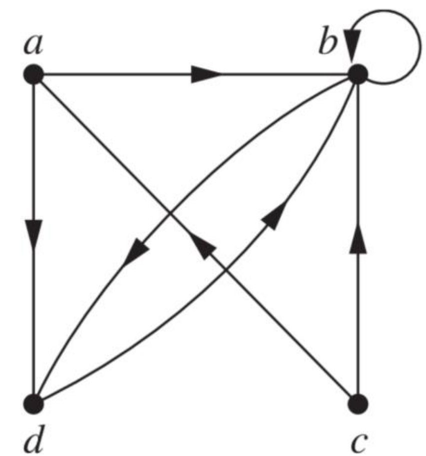
$$M_N = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

Digraphs(关系图)

Defination

A directed graph,or digraph, consists of a set V of vertices (or nodes) together with a set E of ordered pairs of elements of V called edges (or arcs). The vertex a is called the initial vertex of the edge (a, b), and the vertex b is called the terminal vertex of this edge.

The directed graph with vertices a, b, c, and d, and edges $\langle a, b\rangle$, $\langle a, d\rangle$, $\langle b, b\rangle$, $\langle b, d\rangle$, $\langle c, a\rangle$, $\langle c, b\rangle$, and $\langle d, b\rangle$



Digraphs

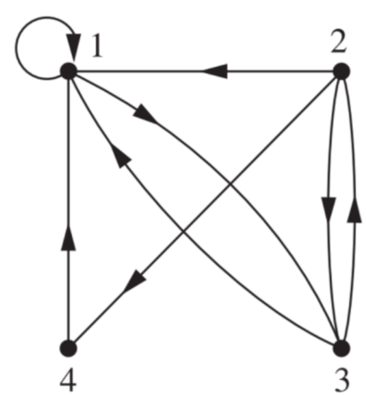
Digraph of Relation R on set A:

- Vertices : Elements of A
- Edges: Ordered pairs in R

The directed graph of the relation

$$R_1 = \{\langle 1,1\rangle, \langle 1,3\rangle, \langle 2,1\rangle, \langle 2,3\rangle, \langle 2,4\rangle, \langle 3,1\rangle, \langle 3,2\rangle, \langle 4,1\rangle\}$$

on the set $\{1,2,3,4\}$ is:



Combining Relations

- Because relations from $\boldsymbol{A}$ to $\boldsymbol{B}$ are subsets of $\boldsymbol{A} \times \boldsymbol{B}$, two relations from $\boldsymbol{A}$ to $\boldsymbol{B}$ can be combined in any way two sets can be combined.

Let $A = \{1, 2, 3\}$. The relations $R_1 = \{\langle 1, 1\rangle, \langle 1, 3\rangle, \langle 2, 1\rangle, \langle 3, 2\rangle\}$ and $R_2 = \{\langle 1, 1\rangle, \langle 1, 3\rangle, \langle 2, 2\rangle, \langle 2, 3\rangle, \langle 3, 1\rangle\}$ can be combined to obtain:

- $R_1 \cup R_2 = \{\langle 1, 1\rangle, \langle 1, 3\rangle, \langle 2, 1\rangle, \langle 2, 2\rangle, \langle 2, 3\rangle, \langle 3, 1\rangle, \langle 3, 2\rangle\}$
- $R_1 \cap R_2 = \{\langle 1, 1\rangle, \langle 1, 3\rangle\}$
- $R_1 - R_2 = \{\langle 2, 1\rangle, \langle 3, 2\rangle\}$
- $R_2 - R_1 = \{\langle 2, 2\rangle, \langle 2, 3\rangle, \langle 3, 1\rangle\}$
- $R_1 \oplus R_2 = \{\langle 2, 1\rangle, \langle 2, 2\rangle, \langle 2, 3\rangle, \langle 3, 1\rangle, \langle 3, 2\rangle\}$

Combining Relations: Relation Matrix

Or represent in matrix:

$$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$,then:

$$M_{R_1 \cup R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \qquad M_{R_1 \cap R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_{R_1 - R_2} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad M_{R_2 - R_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$M_{R_2 \oplus R_1} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Inverse of Relations （关系的逆）

Defination

Let R be a relation from a set A to a set B.The inverse relation of R, denoted by R^{-1}, is the set of ordered pairs $\{\langle b,a\rangle | \langle a,b\rangle \in R\}$ from B to A

Let $A = \{a,b,c\}$, a relation on A, $R = \{\langle a,b\rangle, \langle a,c\rangle, \langle b,a\rangle, \langle c,b\rangle\}$, then:

$$R^{-1} = \{\langle b,a\rangle, \langle c,a\rangle, \langle a,b\rangle, \langle b,c\rangle\}$$

Inverse of Relations: Relation Matrix

R is a relation from $A = \{a_1, a_2, \ldots a_m\}$ to $B = \{b_1, b_2, \ldots b_n\}$,then :

$$M_{R^{-1}} = M_R^T, \text{ where } M_R^T \text{ is the transpose of } M_R$$

Inverse of Relations: Relation Matrix

R is a relation from $A = \{a_1, a_2, \dots a_m\}$ to $B = \{b_1, b_2, \dots b_n\}$,then :

$$M_{R^{-1}} = M_R^T, \text{ where } M_R^T \text{ is the transpose of } M_R$$

Proof:

Assume that $M_R = \langle s_{ij} \rangle_{m \times n}$ where s_{ij} is the element. And $M_{R^{-1}} = \langle t_{ij} \rangle_{n \times m}$, where t_{ij} is the element.

If $s_{ij} = 1$, then we know $\langle a_i, b_j \rangle \in R$, according to the defination of inverse, we have $\langle b_j, a_i \rangle \in R^{-1}$, so $t_{ji} = 1$

If $s_{ij} = 0$, then we know $\langle a_i, b_j \rangle \notin R$. So $\langle b_j, a_i \rangle \notin R^{-1}$, otherwise $\langle a_i, b_j \rangle \in R$, then $t_{ji} = 0$

In any case, we have $s_{ij} = t_{ji}$, and it means M_R and $M_{R^{-1}}$ are transposed each other. Q.E.D

Composite of Relations （关系的合成）

Definition

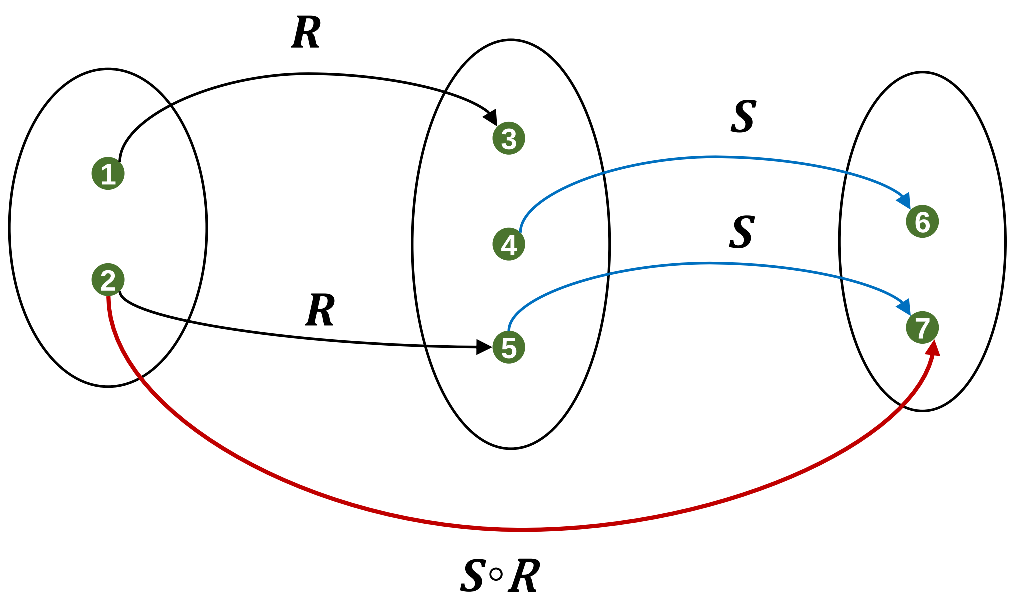
Let R be a relation from A to B, and S be a relation from B to C. The composite of R and S, donate as $S \circ R$, is:

$$S \circ R = \{\langle x, y\rangle | (\exists z)(\langle x, z\rangle \in R \wedge \langle z, y\rangle \in S)\}$$

Let $A = \{1, 2, 3\}$,two relations on A,
$R = \{\langle 1, 1\rangle, \langle 1, 3\rangle, \langle 2, 1\rangle, \langle 2, 2\rangle\}$,$S = \{\langle 1, 2\rangle, \langle 2, 3\rangle, \langle 3, 1\rangle, \langle 3, 3\rangle\}$

Then $S \circ R = \{\langle 1, 1\rangle, \langle 1, 2\rangle, \langle 1, 3\rangle, \langle 2, 2\rangle, \langle 2, 3\rangle\}$

Composite of Relations



Composite of Relations: Relation Matrix

Let A be a set and R and S are two relations on A, the relation matrix of R and S are:

$$M_R = (r_{ij}), M_S = (s_{ij})$$

Then we have:

$$M_{S \circ R} = M_R \odot M_S = (w_{ij})$$

where $\odot$ is the matrix boolean product and $w_{ij} = \vee_{k=1}^{n}(r_{ik} \wedge s_{kj})$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \odot \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} ? \end{pmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 0) \vee (1 \wedge 0) & \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \\ & \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & (1\wedge 1) \vee (0\wedge 0) \vee (1\wedge 1) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ & \end{bmatrix}$$

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Property of Inverse and Composite

1. $dom(R^{-1}) = ran(R)$
2. $ran(R^{-1}) = dom(R)$
3. $(R^{-1})^{-1} = R$
4. $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$
5. $(R \circ S) \circ Q = R \circ (S \circ Q)$
6. $R_1 \circ (R_2 \cup R_3) = R_1 \circ R_2 \cup R_1 \circ R_3$
7. $(R_1 \cup R_2) \circ R_3 = R_1 \circ R_3 \cup R_2 \circ R_3$
8. $R_1 \circ (R_2 \cap R_3) \subseteq R_1 \circ R_2 \cap R_1 \circ R_3$
9. $(R_1 \cap R_2) \circ R_3 \subseteq R_1 \circ R_3 \cap R_2 \circ R_3$

$R_1 \circ (R_2 \cap R_3) \subseteq R_1 \circ R_2 \cap R_1 \circ R_3$

Proof:

$\langle x, y\rangle \in R_1 \circ (R_2 \cap R_3)$

$\Leftrightarrow (\exists z)(\langle x, z\rangle \in (R_2 \cap R_3) \wedge \langle z, y\rangle \in R_1)$

$\Leftrightarrow (\exists z)\langle\langle x, z\rangle \in R_2 \wedge \langle x, z\rangle \in R_3 \wedge \langle z, y\rangle \in R_1\rangle$

$\Rightarrow (\exists z)(\langle x, z\rangle \in R_2 \wedge \langle z, y\rangle \in R_1\rangle \wedge (\exists z)\langle\langle x, z\rangle \in R_3 \wedge \langle z, y\rangle \in R_1)$

$\Leftrightarrow \langle x, y\rangle \in (R_1 \circ R_2) \wedge \langle x, y\rangle \in (R_1 \circ R_3)$

$\Leftrightarrow \langle x, y\rangle \in (R_1 \circ R_2) \cap \langle R_1 \circ R_3\rangle$

Power of Relations （关系的幂）

- For a relation R on set A, $n \in \mathbb{N}$, the nth power of R, donated by R^n, is :
  - $R^0 = I_A$
  - $R^{n+1} = R^n \circ R$
- $R = \{\langle a, b\rangle, \langle b, a\rangle, \langle b, c\rangle, \langle c, d\rangle\}$
  - $R^0 = \{\langle a, a\rangle, \langle b, b\rangle, \langle c, c\rangle, \langle d, d\rangle\}$
  - $R^1 = R = \{\langle a, b\rangle, \langle b, a\rangle, \langle b, c\rangle, \langle c, d\rangle\}$
  - $R^2 = R^4 = \{\langle a, a\rangle, \langle a, c\rangle, \langle b, b\rangle, (b, d)\}$
  - $R^3 = R^5 = \{\langle a, b\rangle, \langle a, d\rangle, \langle b, a\rangle, \langle b, c\rangle\}$

Power of Relations

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  - $R^0 = I_A$
  - $R^{n+1} = R^n \circ R$
- $R = \{\langle a,b\rangle, \langle b,a\rangle, \langle b,c\rangle, \langle c,d\rangle\}$
  - $R^0 = \{\langle a,a\rangle, \langle b,b\rangle, \langle c,c\rangle, \langle d,d\rangle\}$
  - $R^1 = R = \{\langle a,b\rangle, \langle b,a\rangle, \langle b,c\rangle, \langle c,d\rangle\}$
  - $R^2 = R^4 = \{\langle a,a\rangle, \langle a,c\rangle, \langle b,b\rangle, (b,d)\}$
  - $R^3 = R^5 = \{\langle a,b\rangle, \langle a,d\rangle, \langle b,a\rangle, \langle b,c\rangle\}$

Some different power of R are equal.

Power of Relations

$\boldsymbol{A}$ is a finite set and $|\boldsymbol{A}| = n$, $\boldsymbol{R}$ is a relation on $\boldsymbol{A}$, then there exits natural number s, t , $s \neq t$,such that $R^s = R^t$

Proof:

Any relation on $\boldsymbol{A}$ is a subset of $\boldsymbol{A} \times \boldsymbol{A}$, or an element of $\mathcal{P}(A \times A)$.

We know that $|\mathcal{P}(A \times A)| = 2^{|A \times A|} = 2^{|A|^2} = 2^{n^2}$, so there are finite many relations on a finite set $\boldsymbol{A}$. But there are infinite many powers of R. It means there are different powers of $\boldsymbol{R}$ which are equal, or

$$(\exists s)(\exists t)(s \in \mathbb{N} \wedge t \in \mathbb{N} \wedge s \neq t \wedge R^s = R^t)$$

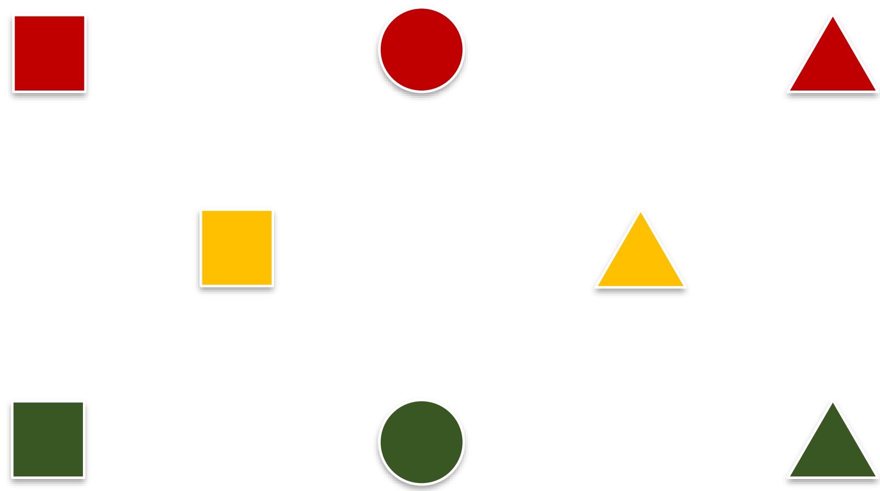
Q.E.D

Power of Relations

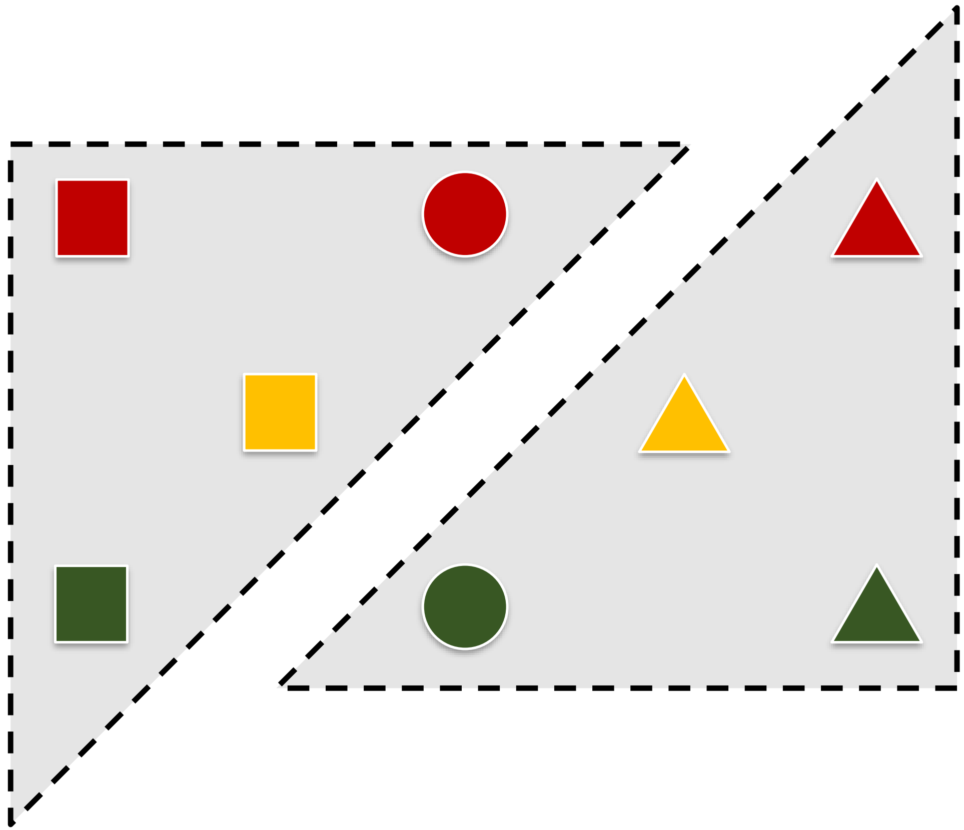
- $R^m \circ R^n = R^{m+n}$
- $(R^m)^n = R^{mn}$
- $\boldsymbol{A}$ is a finite set and $|\boldsymbol{A}| = n$, $\boldsymbol{R}$ is a relation on $\boldsymbol{A}$, and there exits natural number s, t , $s \neq t$,such that $R^s = R^t$
  - $R^{s+k} = R^{t+\mathrm{k}}, k \in \mathbb{N}$
  - $R^{s+kp+i} = R^{t+\mathrm{k}p+i}, k \in \mathbb{N}, i \in \mathbb{N}, p = t - s$
  - Let $B = \{R^0, R^1, \ldots R^{t-1}\}$, then for any natural number q, $R^q \in B$

How can we capture a structure with relations?

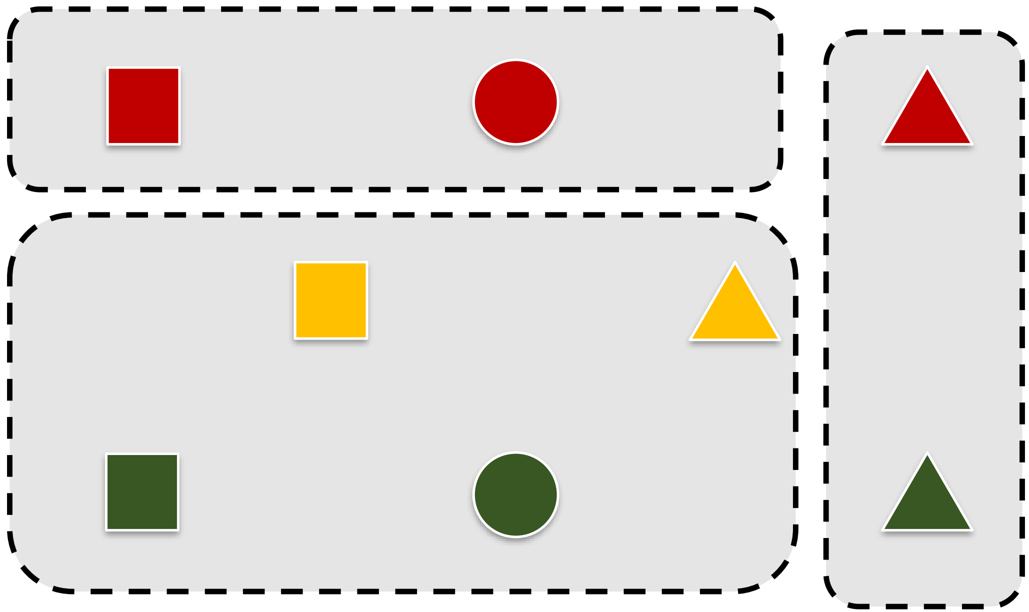
Partitions(划分)



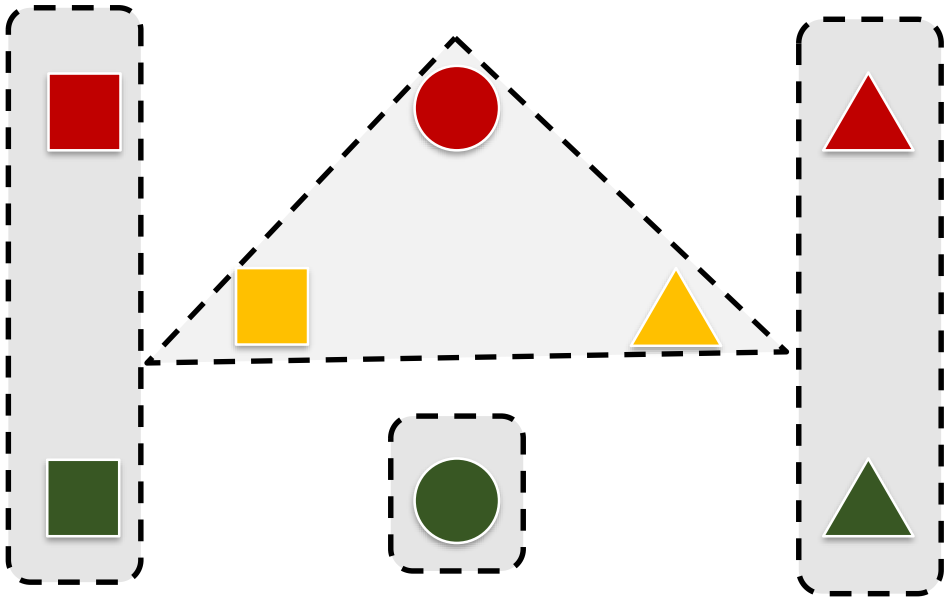
Partitions



Partitions



Partitions



Partitions

- A **partition of a set** is a way of splitting the set into **disjoint, nonempty subsets** so that every element belongs to exactly one subset.
- Intuitively, a partition of a set breaks the set apart into smaller pieces.

Partitions

- More formally, for a non-empty set $\boldsymbol{A}$, a set $\boldsymbol{\pi}$ is called the partition of $\boldsymbol{A}$ if:
  - $\emptyset \notin \boldsymbol{\pi}$
  - $(\forall \boldsymbol{x})(\boldsymbol{x} \in \boldsymbol{\pi} \rightarrow \boldsymbol{x} \subseteq \boldsymbol{A})$
  - $\bigcup \boldsymbol{\pi} = \boldsymbol{A}$
  - $(\forall \boldsymbol{x})(\forall \boldsymbol{y})((\boldsymbol{x} \in \boldsymbol{\pi} \wedge \boldsymbol{y} \in \boldsymbol{\pi} \wedge \boldsymbol{x} \neq \boldsymbol{y}) \rightarrow \boldsymbol{x} \cap \boldsymbol{y} = \emptyset)$

Partitions

- For set $A = \{a, b, c, d\}$
  - $\boldsymbol{\pi_1} = \{\{\boldsymbol{a}\}, \{\boldsymbol{b}, \boldsymbol{c}\}, \{\boldsymbol{d}\}\}$
  - $\boldsymbol{\pi_2} = \{\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}\}\}$
  - $\boldsymbol{\pi_3} = \{\{\boldsymbol{a}, \boldsymbol{b}\}, \{\boldsymbol{c}\}, \{\boldsymbol{a}, \boldsymbol{d}\}\}$
  - $\boldsymbol{\pi_4} = \{\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}\}\}$

valid partitions

not partition

What is the relationship between partitions and relations?

Relation in Partition

- For a partition $\boldsymbol{\pi}$ of set $\boldsymbol{A}$, we can define a relation R such that $\boldsymbol{aRb}$ if $\boldsymbol{a}$ and $\boldsymbol{b}$ are in a same partition block.

$$\boldsymbol{R = \{\langle a, b\rangle | (\exists \pi_0)(\pi_0 \in \pi \land a \in \pi_0 \land b \in \pi_0)\}}$$

- Then we have:
  - $\boldsymbol{aRa}$ **for** $\forall \boldsymbol{a} \in \boldsymbol{A}$
  - If $\boldsymbol{aRb}$, then $\boldsymbol{bRa}$
  - If $\boldsymbol{aRb}$ and $\boldsymbol{bRc}$, then $\boldsymbol{aRc}$

Relation in Partition

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$$\boldsymbol{R} = \{\langle \boldsymbol{a}, \boldsymbol{b} \rangle | (\exists \boldsymbol{\pi_0})(\boldsymbol{\pi_0} \in \boldsymbol{\pi} \wedge \boldsymbol{a} \in \boldsymbol{\pi_0} \wedge \boldsymbol{b} \in \boldsymbol{\pi_0})\}$$

- Then we have:
  - $\boldsymbol{aRa}$ **for** $\forall \boldsymbol{a} \in \boldsymbol{A}$

    An element is in the same partition block with itself.
  - If $\boldsymbol{aRb}$, then $\boldsymbol{bRa}$

    If **a** and **b** are in the same block, then **b** and **b** are in the same block too.
  - If $\boldsymbol{aRb}$ and $\boldsymbol{bRc}$, then $\boldsymbol{aRc}$

    If **a** and **b** are in the same block, **b** and **c** are also in the same block, then **a** and **c** are in the same block which contains **b**.

Relation in Partition

$$(\forall a)(a \in A \rightarrow aRa)$$

$$(\forall a)(\forall b)((a \in A \land b \in A \land aRb) \rightarrow bRa)$$

$$(\forall a)(\forall b)(\forall c)$$
$$\big((a \in A \land b \in A \land c \in A \land aRb \land bRc) \rightarrow aRc\big)$$

Relation in Partition

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$$((a \in A \land b \in A \land c \in A \land aRb \land bRc) \rightarrow aRc)$$

Reflexivity(自反性)

- Some relations always hold from any element to itself.
- Examples:
  - $\boldsymbol{x} = \boldsymbol{x}$ for any x
  - $\boldsymbol{A} \subseteq \boldsymbol{A}$ for any set A
  - $\boldsymbol{x} \geq \boldsymbol{x}$ for any x
- Relations of this sort are called reflexive.

$$\boldsymbol{R} \textbf{ on } \boldsymbol{A} \textbf{ is reflexive} \Leftrightarrow (\forall \boldsymbol{a})(\boldsymbol{a} \in \boldsymbol{A} \rightarrow \boldsymbol{aRa})$$

Reflexivity: Example

- $\boldsymbol{I_A}$ and $\boldsymbol{E_A}$
- $=, \geq, \leq$ on $\mathbb{R}$
- $\boldsymbol{R} = \{\langle 1,1\rangle, \langle 2,2\rangle, \langle 3,1\rangle\}$ on set $\boldsymbol{A} = \{1,2,3\}$

Reflexivity: Example

- $\boldsymbol{I_A}$ and $\boldsymbol{E_A}$ ☺
  - aI_Aa and aE_Aa for all $a \in A$
- $=, \geq, \leq$ on $\mathbb{R}$ ☺
  - $a = a, a \geq a, a \leq a$ for all $a \in \mathbb{R}$
- $\boldsymbol{R} = \{\langle 1,1\rangle, \langle 2,2\rangle, \langle 3,1\rangle\}$ on set $\boldsymbol{A} = \{1,2,3\}$ ☹
  - $\langle 3,3\rangle \notin R$, it's not reflexive

Relation in Partition

$$(\forall a)(a \in A \rightarrow aRa)$$

$$(\forall a)(\forall b)((a \in A \wedge b \in A \wedge aRb) \rightarrow bRa)$$

$$(\forall a)(\forall b)(\forall c)$$
$$((a \in A \wedge b \in A \wedge c \in A \wedge aRb \wedge bRc) \rightarrow aRc)$$

Symmetry(对称性)

- In some relations, the relative order of the objects doesn't matter.
- Examples:
  - If $\boldsymbol{x} = \boldsymbol{y}$ then $\boldsymbol{y} = \boldsymbol{x}$
  - If $\boldsymbol{x} \equiv_{\boldsymbol{k}} \boldsymbol{y}$ then $\boldsymbol{y} \equiv_{\boldsymbol{k}} \boldsymbol{x}$
- These relations are called symmetric.

$\boldsymbol{R}$ on $\boldsymbol{A}$ is symmetric $\Leftrightarrow$

$$(\forall \boldsymbol{a})(\forall \boldsymbol{b})((\boldsymbol{a} \in \boldsymbol{A} \wedge \boldsymbol{b} \in \boldsymbol{A} \wedge \boldsymbol{aRb}) \rightarrow \boldsymbol{bRa})$$

Symmetry : Example

- $\boldsymbol{I_A}$ and $\boldsymbol{E_A}$
- $\boldsymbol{N_A}$
- $\boldsymbol{R} = \{\langle 1,2\rangle, \langle 2,1\rangle, \langle 3,3\rangle\}$ on $\{1,2,3\}$

Symmetry : Example

- $\boldsymbol{I_A}$ and $\boldsymbol{E_A}$
  - $aI_Ab \rightarrow bI_Aa$, $aE_Ab \rightarrow bE_Aa$
- $\boldsymbol{N_A}$
  - $aN_Ab \rightarrow bN_Aa$ always hold cause aN_Ab never holds
- $\boldsymbol{R} = \{\langle 1,2\rangle, \langle 2,1\rangle, \langle 3,3\rangle\}$ on $\{1,2,3\}$

Symmetry : Relation Matrix

$\boldsymbol{R}$ is symmetric $\Leftrightarrow \boldsymbol{M_R}$ is symmetric matrix

- So if $\boldsymbol{R}$ is symmetric , then $\boldsymbol{R^{-1} = R}$ as $\boldsymbol{M_{R^{-1}} = M_R^T = M_R}$
- $\boldsymbol{M_{E_A}}$, $\boldsymbol{M_{I_A}}$ and $\boldsymbol{M_{N_A}}$ are both symmetric matrix

Relation in Partition

$$(\forall a)(a \in A \rightarrow aRa)$$

$$(\forall a)(\forall b)((a \in A \wedge b \in A \wedge aRb) \rightarrow bRa)$$

$$(\forall a)(\forall b)(\forall c)\Big((a \in A \wedge b \in A \wedge c \in A \wedge aRb \wedge bRc) \rightarrow aRc\Big)$$

Transitivity(传递性)

- Many relations can be chained together.
- Examples:
  - If $x = y$ and $y = z$ then $x = z$
  - If $P \subseteq Q$ and $Q \subseteq S$, then $P \subseteq S$
  - If $x \equiv_k y$ and $y \equiv_k z$, then $x \equiv_k z$
- These relations are called transitive.

$$R \text{ on } A \text{ is transitive} \Leftrightarrow$$

$$(\forall a)(\forall b)(\forall c)\big((a \in A \wedge b \in A \wedge c \in A \wedge aRb \wedge bRc) \rightarrow aRc\big)$$

Transitivity : Example

- $\boldsymbol{I_A}$ and $\boldsymbol{E_A}$

- $\boldsymbol{N_A}$

- $\boldsymbol{R} = \{\langle 1,2\rangle, \langle 2,3\rangle, \langle 3,3\rangle\}$ on $\{1,2,3\}$

Transitivity : Example

- $\boldsymbol{I_A}$ and $\boldsymbol{E_A}$

- $\boldsymbol{N_A}$

- $\boldsymbol{R} = \{\langle 1,2\rangle, \langle 2,3\rangle, \langle 3,3\rangle\}$ on $\{1,2,3\}$
  - $\langle 1,2\rangle \in R, \langle 2,3\rangle \in R$ but $\langle 1,3\rangle \notin R$

Reflexivity, Symmetry and Transitivity

- If $\boldsymbol{R_1}, \boldsymbol{R_2}$ are reflexive, then $\boldsymbol{R_1^{-1}}, \boldsymbol{R_1} \cup \boldsymbol{R_2}, \boldsymbol{R_1} \cap \boldsymbol{R_2}$ are also reflexive

- If $\boldsymbol{R_1}, \boldsymbol{R_2}$ are symmetric, then $\boldsymbol{R_1^{-1}}, \boldsymbol{R_1} \cup \boldsymbol{R_2}, \boldsymbol{R_1} \cap \boldsymbol{R_2}$ are also symmetric

- If $\boldsymbol{R_1}, \boldsymbol{R_2}$ are transitive, then $\boldsymbol{R_1^{-1}}, \boldsymbol{R_1} \cap \boldsymbol{R_2}$ are also transitive

Reflexivity, Symmetry and Transitivity

- If R_1, R_2 are symmetric, then $R_1 \cup R_2$ are also symmetric.

Proof:

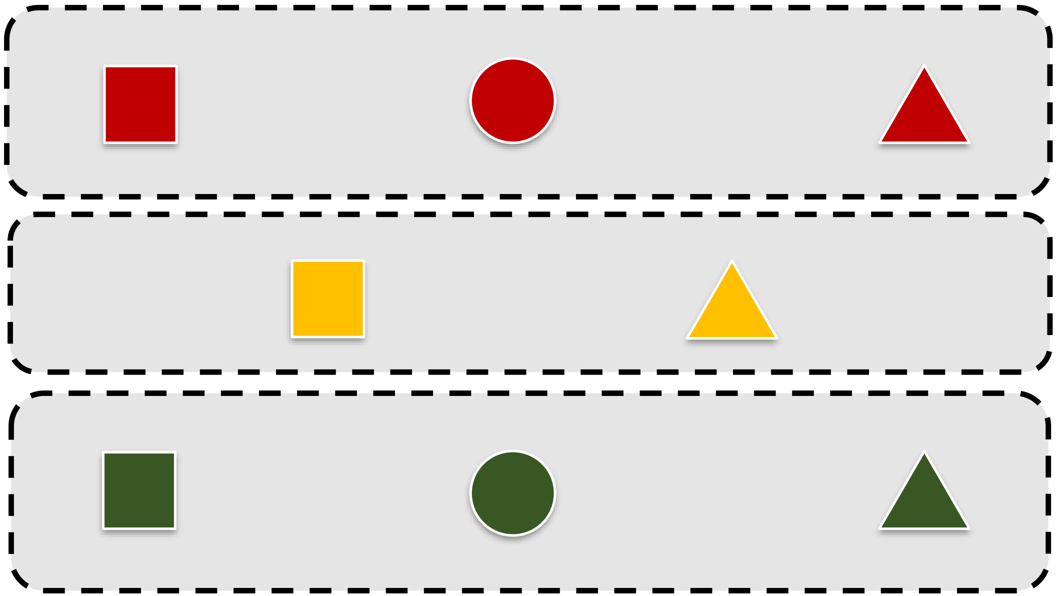
$\langle x, y \rangle \in R_1 \cup R_2$
$\Leftrightarrow \langle x, y \rangle \in R_1 \vee \langle x, y \rangle \in R_2$
$\Leftrightarrow \langle y, x \rangle \in R_1 \vee \langle y, x \rangle \in R_2$
$\Leftrightarrow \langle y, x \rangle \in R_1 \cup R_2$

Q.E.D

Equivalence Relations(等价关系)

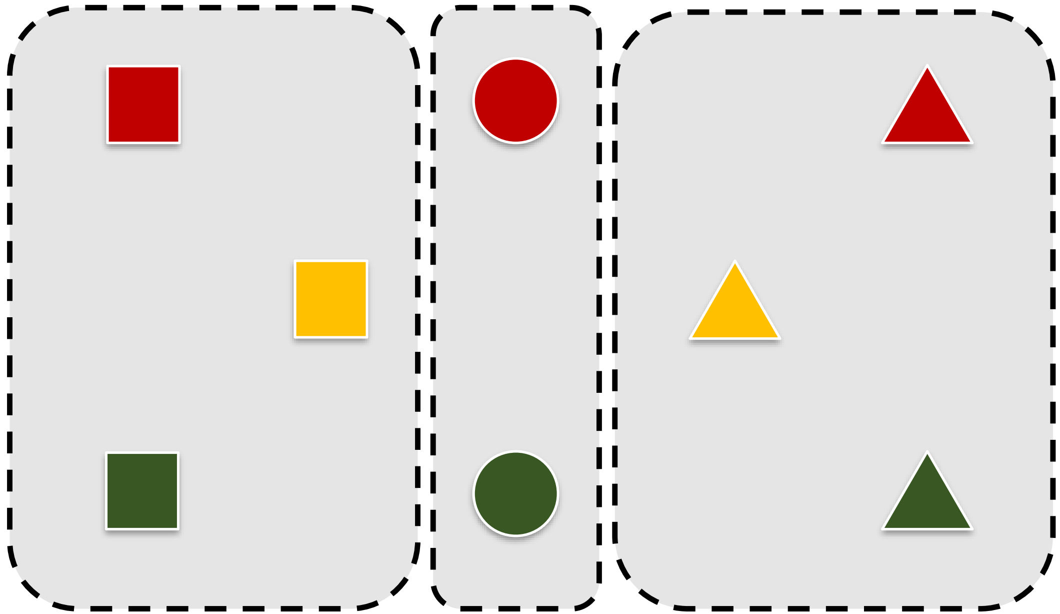
- An **equivalence relation** is a relation that is **reflexive, symmetric and transitive**.
- Some examples:
  - $x = y$
  - $x \equiv_k y$
  - x has the same color with y
  - x has the same shape with y

Partitions



aRb if a has the color with b

Partitions



aRb if a has the **shape** with b

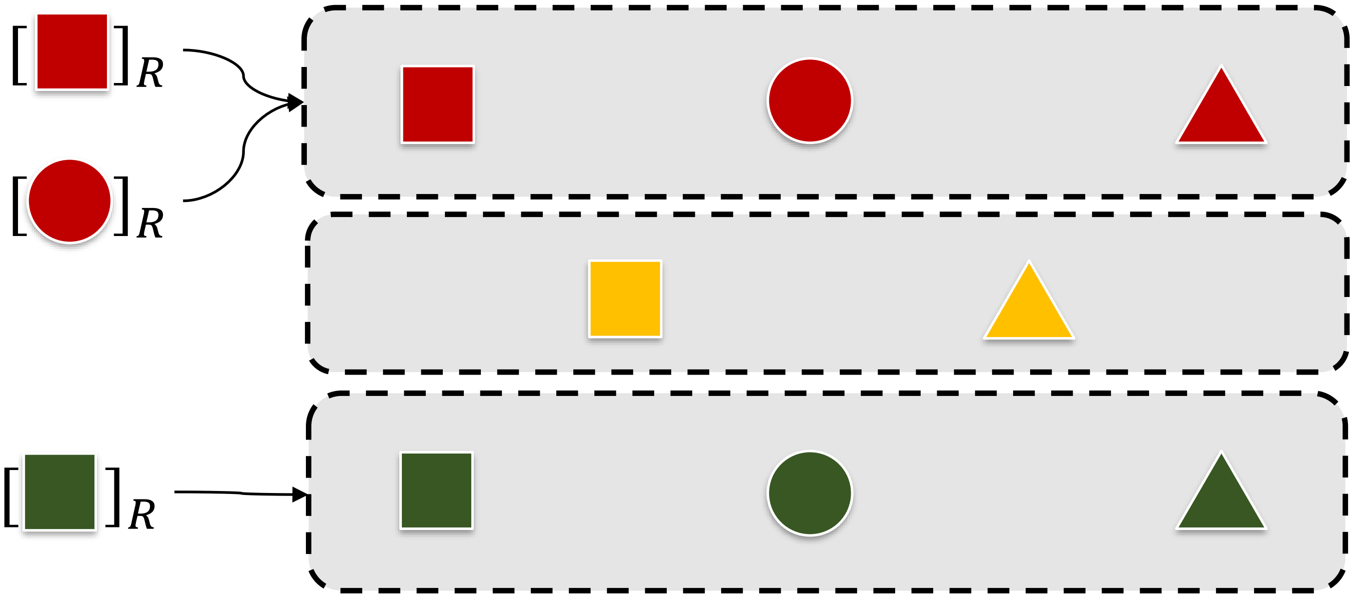
Equivalence Classes(等价类)

- Given an equivalence relation $\boldsymbol{R}$ over a set $\boldsymbol{A}$, for any $\boldsymbol{x} \in \boldsymbol{A}$, the **equivalence class of $\boldsymbol{x}$** is the set:

$$[\boldsymbol{x}]_{\boldsymbol{R}} = \{\boldsymbol{y} | \boldsymbol{y} \in \boldsymbol{A} \wedge \boldsymbol{xRy}\}$$

- Intuitively, the set $[\boldsymbol{x}]_{\boldsymbol{R}}$ contains all elements of $\boldsymbol{A}$ that are related to $\boldsymbol{x}$ by relation $\boldsymbol{R}$.

Partitions



aRb if a has the ***color*** *with b*

Quotient Set(商集)

- Given an equivalence relation $\boldsymbol{R}$ over a set $\boldsymbol{A}$, the quotient set of $\boldsymbol{A}$ is set of all the disjoint equivalence classes of $\boldsymbol{A}$, donated by $\boldsymbol{A/R}$**:**

$$\boldsymbol{A/R} = \{\boldsymbol{y} | (\exists \boldsymbol{x})(\boldsymbol{x} \in \boldsymbol{A} \wedge \boldsymbol{y} = [\boldsymbol{x}]_{\boldsymbol{R}})\}$$

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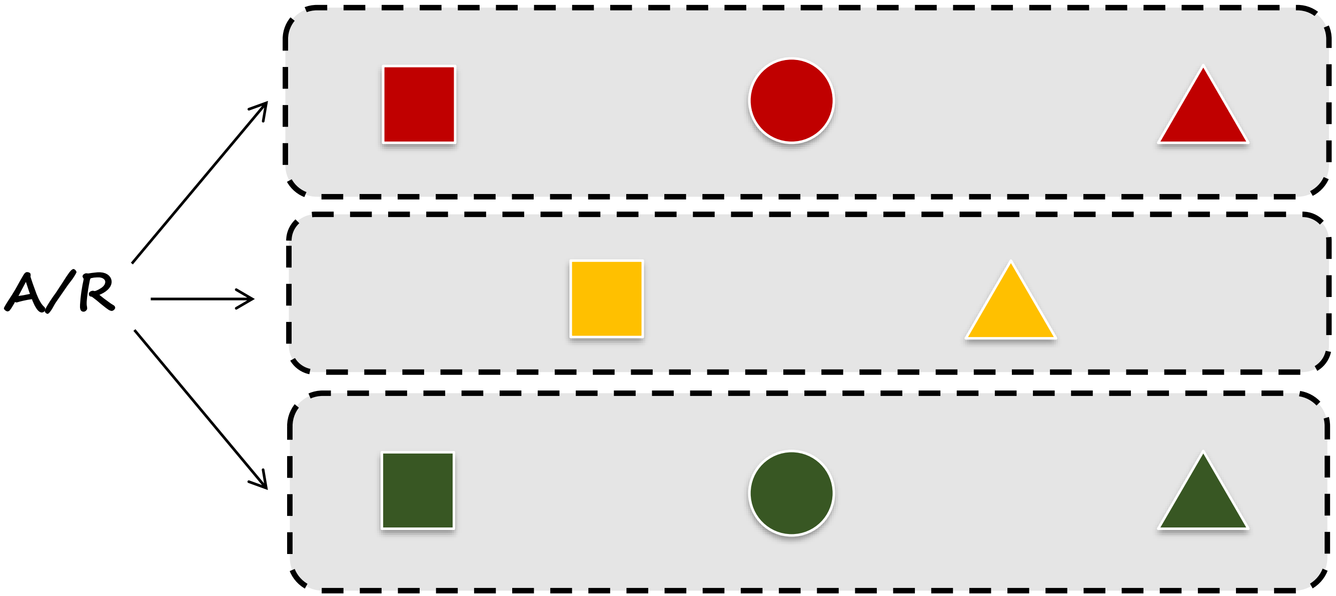
$$\boldsymbol{A}/\boldsymbol{R} = \{\boldsymbol{y} | (\exists \boldsymbol{x})(\boldsymbol{x} \in \boldsymbol{A} \wedge \boldsymbol{y} = [\boldsymbol{x}]_{\boldsymbol{R}})\}$$

- For $\boldsymbol{R} = \{\langle x, y\rangle | x \equiv_3 y\}$ on set $\boldsymbol{A} = \{1,2,3 \ldots 8\}$
  - $\boldsymbol{A}/\boldsymbol{R} = \{[1]_{\boldsymbol{R}}, [2]_{\boldsymbol{R}}, [3]_{\boldsymbol{R}}\} = \{\{1,4,7\}, \{2,5,8\}, \{3,6\}\}$

Partitions



aRb if a has the color with b

Quotient Set and Partition

Given an equivalence relation $\boldsymbol{R}$ over a set $\boldsymbol{A}$, then $\boldsymbol{A}/\boldsymbol{R}$ is a partition of $\boldsymbol{A}$

Quotient Set and Partition

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For any $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{A}, [\boldsymbol{x}]_{\boldsymbol{R}}, [\boldsymbol{y}]_{\boldsymbol{R}} \in \boldsymbol{A}/\boldsymbol{R}$:

Definition of Partition

$\emptyset \notin \boldsymbol{A}/\boldsymbol{R}$

$(\forall \boldsymbol{\pi_0})(\boldsymbol{\pi_0} \in \boldsymbol{A}/\boldsymbol{R} \to \boldsymbol{\pi_0} \subseteq \boldsymbol{A})$

$\cup \boldsymbol{A}/\boldsymbol{R} = \boldsymbol{A}$

$\langle \forall \boldsymbol{x} \rangle \langle \forall \boldsymbol{y} \rangle \langle (\boldsymbol{x} \in \boldsymbol{A}/\boldsymbol{R} \wedge \boldsymbol{y} \in \boldsymbol{A}/\boldsymbol{R} \wedge \boldsymbol{x} \neq \boldsymbol{y}) \to \boldsymbol{x} \cap \boldsymbol{y} = \emptyset \rangle$

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For any $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{A}, [\boldsymbol{x}]_{\boldsymbol{R}}, [\boldsymbol{y}]_{\boldsymbol{R}} \in \boldsymbol{A}/\boldsymbol{R}$:

$[\boldsymbol{x}]_{\boldsymbol{R}} \neq \emptyset$ because $\boldsymbol{x} \in [\boldsymbol{x}]_{\boldsymbol{R}}$ → $\emptyset \notin \boldsymbol{A}/\boldsymbol{R}$

Definition of Partition

$\emptyset \notin \boldsymbol{A}/\boldsymbol{R}$

$(\forall \boldsymbol{\pi_0})(\boldsymbol{\pi_0} \in \boldsymbol{A}/\boldsymbol{R} \rightarrow \boldsymbol{\pi_0} \subseteq \boldsymbol{A})$

$\bigcup \boldsymbol{A}/\boldsymbol{R} = \boldsymbol{A}$

$\langle \forall \boldsymbol{x} \rangle \langle \forall \boldsymbol{y} \rangle \langle (\boldsymbol{x} \in \boldsymbol{A}/\boldsymbol{R} \wedge \boldsymbol{y} \in \boldsymbol{A}/\boldsymbol{R} \wedge \boldsymbol{x} \neq \boldsymbol{y}) \rightarrow \boldsymbol{x} \cap \boldsymbol{y} = \emptyset \rangle$

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$[\boldsymbol{x}]_{\boldsymbol{R}} \neq \emptyset$ cause $\boldsymbol{x} \in [\boldsymbol{x}]_{\boldsymbol{R}}$ → $\emptyset \notin \boldsymbol{A}/\boldsymbol{R}$

$[\boldsymbol{x}]_{\boldsymbol{R}} \subseteq \boldsymbol{A}$ → $(\forall \boldsymbol{\pi_0})(\boldsymbol{\pi_0} \in \boldsymbol{A}/\boldsymbol{R} \to \boldsymbol{\pi_0} \subseteq \boldsymbol{A})$

Definition of Partition

$\emptyset \notin \boldsymbol{A}/\boldsymbol{R}$

$(\forall \boldsymbol{\pi_0})(\boldsymbol{\pi_0} \in \boldsymbol{A}/\boldsymbol{R} \to \boldsymbol{\pi_0} \subseteq \boldsymbol{A})$

$\bigcup \boldsymbol{A}/\boldsymbol{R} = \boldsymbol{A}$

$\langle \forall \boldsymbol{x} \rangle \langle \forall \boldsymbol{y} \rangle \langle (\boldsymbol{x} \in \boldsymbol{A}/\boldsymbol{R} \wedge \boldsymbol{y} \in \boldsymbol{A}/\boldsymbol{R} \wedge \boldsymbol{x} \neq \boldsymbol{y}) \to \boldsymbol{x} \cap \boldsymbol{y} = \emptyset \rangle$

Quotient Set and Partition

Given an equivalence relation $\boldsymbol{R}$ over a set $\boldsymbol{A}$, then $\boldsymbol{A/R}$ is a partition of $\boldsymbol{A}$

Proof:

For any $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{A}$, $[\boldsymbol{x}]_{\boldsymbol{R}}, [\boldsymbol{y}]_{\boldsymbol{R}} \in \boldsymbol{A/R}$:

Definition of Partition

$[\boldsymbol{x}]_{\boldsymbol{R}} \neq \emptyset$ cause $\boldsymbol{x} \in [\boldsymbol{x}]_{\boldsymbol{R}}$ $\rightarrow$ $\emptyset \notin \boldsymbol{A/R}$

$[\boldsymbol{x}]_{\boldsymbol{R}} \subseteq \boldsymbol{A}$ $\rightarrow$ $(\forall \boldsymbol{\pi_0})(\boldsymbol{\pi_0} \in \boldsymbol{A/R} \rightarrow \boldsymbol{\pi_0} \subseteq \boldsymbol{A})$

$\boldsymbol{x} \in [\boldsymbol{x}]_{\boldsymbol{R}}$ $\rightarrow$ $\bigcup \boldsymbol{A/R} = \boldsymbol{A}$

$\langle \forall \boldsymbol{x} \rangle \langle \forall \boldsymbol{y} \rangle \langle (\boldsymbol{x} \in \boldsymbol{A/R} \land \boldsymbol{y} \in \boldsymbol{A/R} \land \boldsymbol{x} \neq \boldsymbol{y}) \rightarrow \boldsymbol{x} \cap \boldsymbol{y} = \emptyset \rangle$

Quotient Set and Partition

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Proof:

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Definition of Partition

$[\boldsymbol{x}]_{\boldsymbol{R}} \neq \emptyset$ cause $\boldsymbol{x} \in [\boldsymbol{x}]_{\boldsymbol{R}}$ → $\emptyset \notin \boldsymbol{A/R}$

$[\boldsymbol{x}]_{\boldsymbol{R}} \subseteq \boldsymbol{A}$ → $(\forall \boldsymbol{\pi_0})(\boldsymbol{\pi_0} \in \boldsymbol{A/R} \rightarrow \boldsymbol{\pi_0} \subseteq \boldsymbol{A})$

$\boldsymbol{x} \in [\boldsymbol{x}]_{\boldsymbol{R}}$ → $\bigcup \boldsymbol{A/R} = \boldsymbol{A}$

$[\boldsymbol{x}]_{\boldsymbol{R}} = [\boldsymbol{y}]_{\boldsymbol{R}}$ or $[\boldsymbol{x}]_{\boldsymbol{R}} \cap [\boldsymbol{y}]_{\boldsymbol{R}} = \emptyset$ → $\langle \forall \boldsymbol{x} \rangle \langle \forall \boldsymbol{y} \rangle \langle (\boldsymbol{x} \in \boldsymbol{A/R} \wedge \boldsymbol{y} \in \boldsymbol{A/R} \wedge \boldsymbol{x} \neq \boldsymbol{y}) \rightarrow \boldsymbol{x} \cap \boldsymbol{y} = \emptyset \rangle$

Equivalence Relation and Partition

- For a partition $\boldsymbol{\pi}$ of set $\boldsymbol{A}$, we can define a relation R such that $\boldsymbol{aRb}$ if $\boldsymbol{a}$ and $\boldsymbol{b}$ are in a same partition.

$$\boldsymbol{R} = \{\langle \boldsymbol{a}, \boldsymbol{b} \rangle | \boldsymbol{\pi_0} \in \boldsymbol{\pi} \wedge \boldsymbol{a} \in \boldsymbol{\pi_0} \wedge \boldsymbol{b} \in \boldsymbol{\pi_0}\}$$

A partition can generate an equivalence relation.

- Given an equivalence relation $\boldsymbol{R}$ over a set $\boldsymbol{A}$, then $\boldsymbol{A}/\boldsymbol{R}$ is a partition of $\boldsymbol{A}$

An equivalence relation can generate a partition.

Equivalence relations precisely capture what it means to be a partition.

Binary relations give us a **common language** to describe **common structures**.

Equivalence Relation In Real Life

- Most modern programming languages include some sort of hash table data structure.
  - **Java: HashMap**
  - **C++: std::unordered_map**
  - **Python: dict**
- If you insert a key/value pair and then try to look up a key, the implementation has to be able to tell whether two keys are equal.
- Although each language has a different mechanism for specifying this, many languages describe them in similar ways...

Equivalence Relation In Real Life

- “The equals method implements an equivalence relation on non-null object references:
  - It is reflexive: for any non-null reference value x, x.equals⟨x⟩ should return true.
  - It is symmetric: for any non-null reference values x and y, x.equals⟨y⟩ should return true if and only if y.equals⟨x⟩ returns true.
  - It is transitive: for any non-null reference values x, y, and z, if x.equals⟨y⟩ returns true and y.equals⟨z⟩ returns true, then x.equals⟨z⟩ should return true.”

Java 8 Documentation

Equivalence Relation In Real Life

- “The equals method implements an equivalence relation on non-null object references:
  - It is reflexive: for any non-null reference value x, x.equals⟨x⟩ should return true.
  - It is symmetric: for any non-null reference values x and y, x.equals⟨y⟩ should return true if and only if y.equals⟨x⟩ returns true.
  - It is transitive: for any non-null reference values x, y, and z, if x.equals⟨y⟩ returns true and y.equals⟨z⟩ returns true, then x.equals⟨z⟩ should return true.”

Java 8 Documentation

Equivalence Relation In Real Life

“Each unordered associative container is parameterized by *Key*, by a function object type *Hash* that meets the Hash requirements ⟨17.6.3.4⟩ and acts as a hash function for argument values of type *Key*, and by a binary predicate Pred that induces an equivalence relation on values of type *Key*. Additionally, unordered_map and unordered_multimap associate an arbitrary mapped type T with the *Key*.”

C++14 ISO Spec, §23.2.5/3

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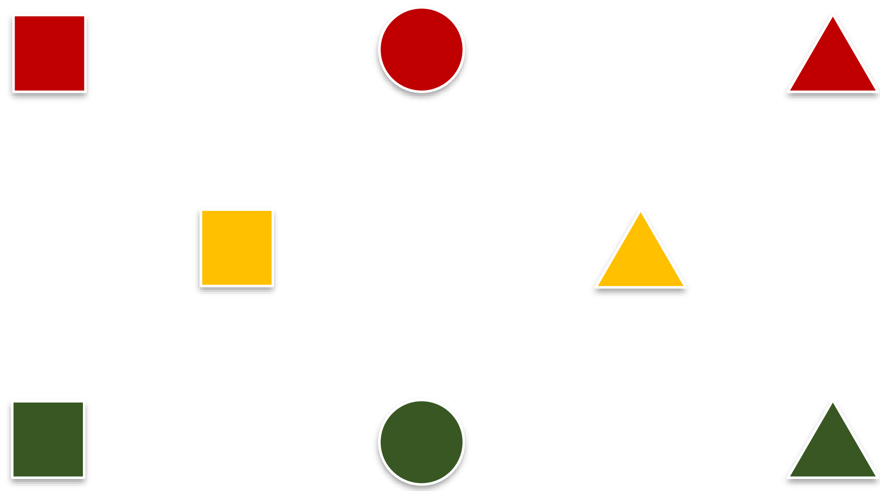
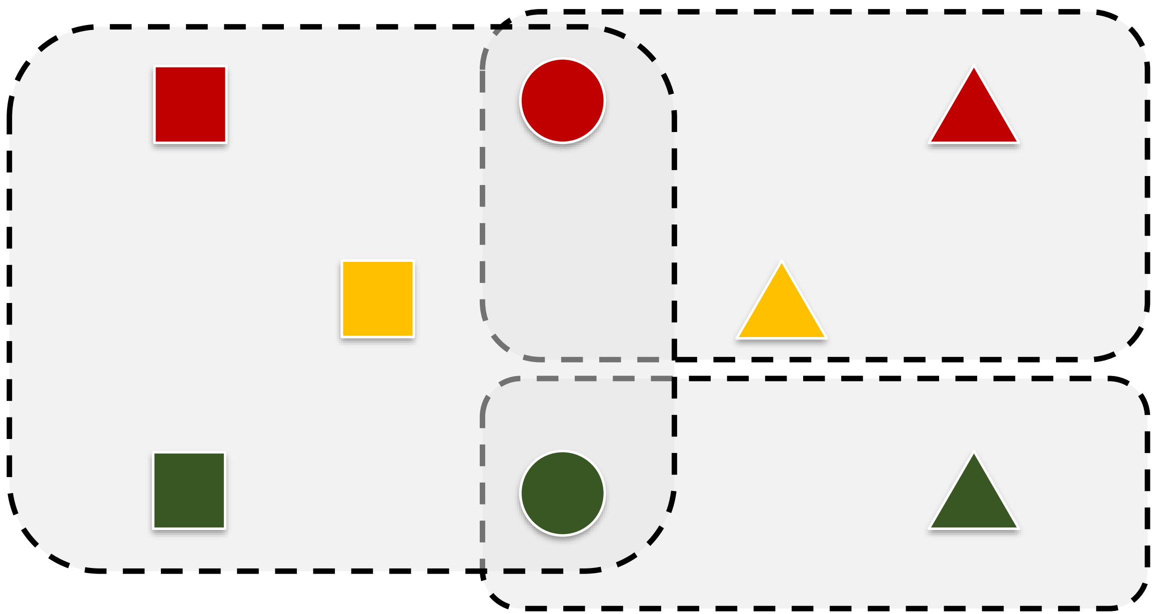
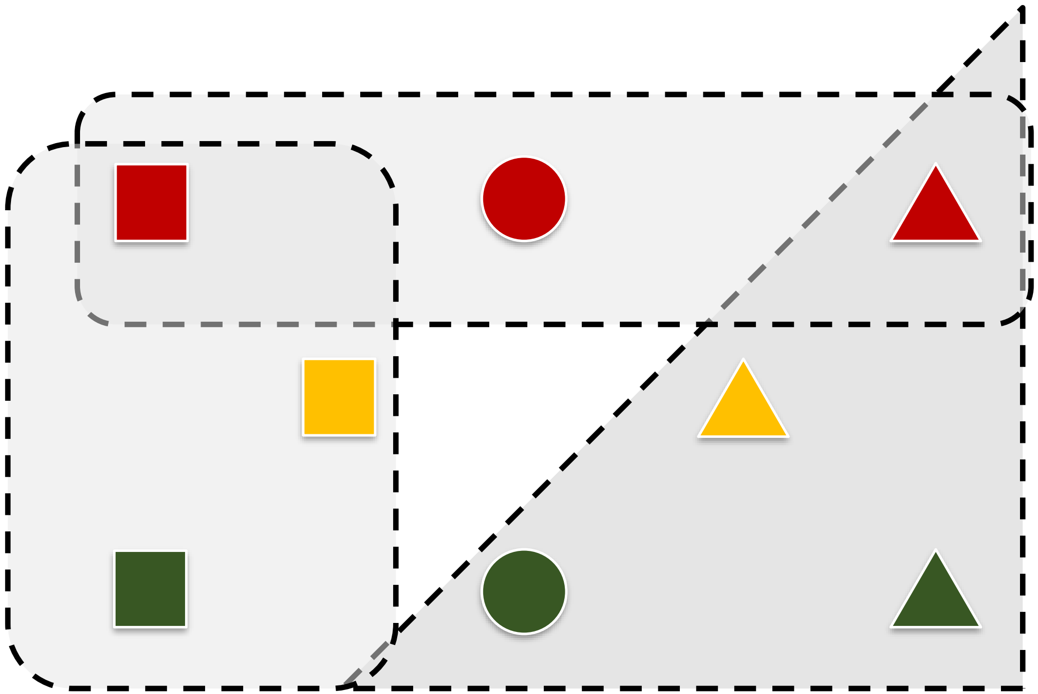
Covers（覆盖）



Covers



Covers



Covers

- A **cover of a set** is a way of splitting the set into **nonempty subsets** so that every element belongs to at least one subset.
- Intuitively, a cover contains pieces that cover every element of a set.
- **A partition is a cover, but a cover is not a partition.**

Covers

- More formally, for a non-empty set $\boldsymbol{A}$, a set $\boldsymbol{C}$ is called the cover of $\boldsymbol{A}$ if:
  - $\emptyset \notin \boldsymbol{C}$
  - $(\forall \boldsymbol{x})(\boldsymbol{x} \in \boldsymbol{C} \rightarrow \boldsymbol{x} \subseteq \boldsymbol{A})$
  - $\bigcup \boldsymbol{C} = \boldsymbol{A}$
  - $\cancel{(\forall \boldsymbol{x})(\forall \boldsymbol{y})((\boldsymbol{x} \in \boldsymbol{C} \land \boldsymbol{y} \in \boldsymbol{C} \land \boldsymbol{x} \neq \boldsymbol{y}) \rightarrow \boldsymbol{x} \cap \boldsymbol{y} = \emptyset)}$

Covers

- For set $A = \{a, b, c, d\}$
  - $\boldsymbol{C_1 = \{\{a\}, \{b, c\}, \{d\}\}}$
  - $\boldsymbol{C_2 = \{\{a, b, c, d\}\}}$
  - $\boldsymbol{C_3 = \{\{a, b\}, \{c\}, \{a, d\}\}}$
  - $\boldsymbol{C_4 = \{\{a, b, d\}\}}$

valid covers

not a cover

Relation in Cover

- For a cover $\boldsymbol{C}$ of set $\boldsymbol{A}$, we can define a relation $\boldsymbol{R}$ such that $\boldsymbol{aRb}$ if $\boldsymbol{a}$ and $\boldsymbol{b}$ are in a same cover block.

$$\boldsymbol{R} = \{\langle \boldsymbol{a}, \boldsymbol{b} \rangle | (\exists \boldsymbol{C_0})(\boldsymbol{C_0} \in \boldsymbol{C} \wedge \boldsymbol{a} \in \boldsymbol{C_0} \wedge \boldsymbol{b} \in \boldsymbol{C_0})\}$$

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- Then we have:
  - $\boldsymbol{aRa}$ for all $\boldsymbol{a} \in \boldsymbol{A}$
  - If $\boldsymbol{aRb}$, then $\boldsymbol{bRa}$
  - ~~If $\boldsymbol{aRb}$ and $\boldsymbol{bRc}$, then $\boldsymbol{aRc}$~~

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- Then we have:
  - $\boldsymbol{aRa}$ for all $\boldsymbol{a} \in \boldsymbol{A}$

    An element is in the same block with itself.

  - If $\boldsymbol{aRb}$, then $\boldsymbol{bRa}$

    If **a** and **b** are in the same block, then **b** and **b** are in the same block too.

  - ~~If $\boldsymbol{aRb}$ and $\boldsymbol{bRc}$, then $\boldsymbol{aRc}$~~

    If **a** and **b** are in the same block C_0, **b** and **c** are also in the same block $C_1 \neq C_0$, **a** and **b** are not in the same block.

Relation in Cover

- For a cover $\boldsymbol{C}$ of set $\boldsymbol{A}$, we can define a relation $\boldsymbol{R}$ such that $\boldsymbol{aRb}$ if $\boldsymbol{a}$ and $\boldsymbol{b}$ are in a same cover block.

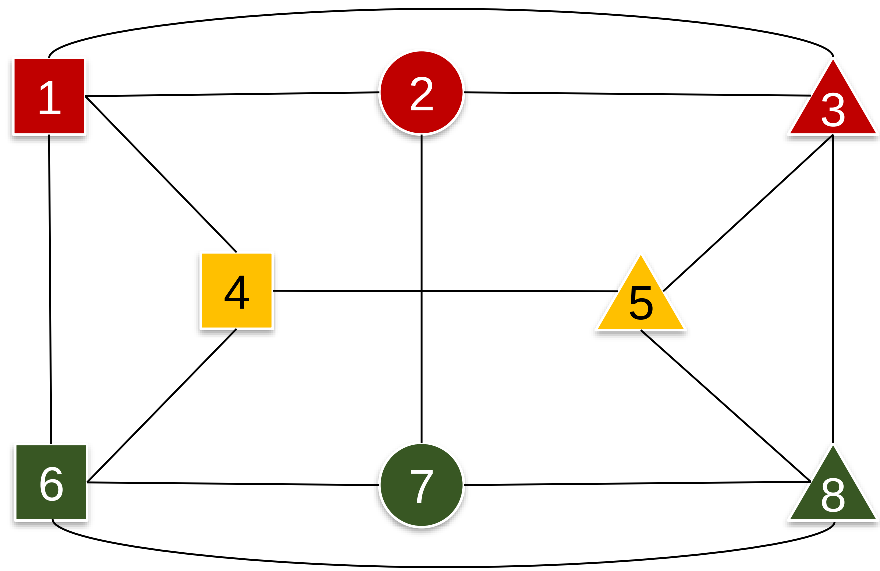
$$\boldsymbol{R} = \{\langle \boldsymbol{a}, \boldsymbol{b} \rangle | (\exists \boldsymbol{C_0})(\boldsymbol{C_0} \in \boldsymbol{C} \land \boldsymbol{a} \in \boldsymbol{C_0} \land \boldsymbol{b} \in \boldsymbol{C_0})\}$$

- Then we have:
  - $\boldsymbol{aRa}$ for all $\boldsymbol{a} \in \boldsymbol{A}$ Reflexivity
  - If $\boldsymbol{aRb}$, then $\boldsymbol{bRa}$ Symmetry
  - ~~If $\boldsymbol{aRb}$ and $\boldsymbol{bRc}$, then $\boldsymbol{aRc}$~~

Compatible Relations (相容关系)

- An **compatible relation** is a relation that is **reflexive and symmetric**.
- Some examples:
  - x has at least one same letter with y
  - $x \cap y \neq \emptyset$
  - x has same color or same shape with y

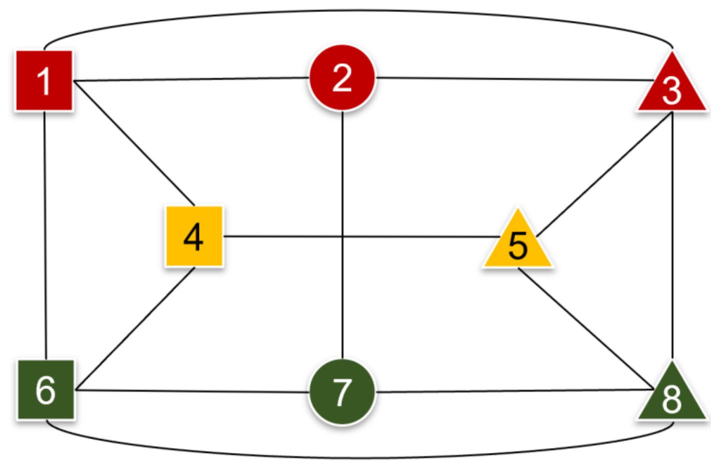
Compatible Relation



*xRy if x has same **color** or same **shape** with y*

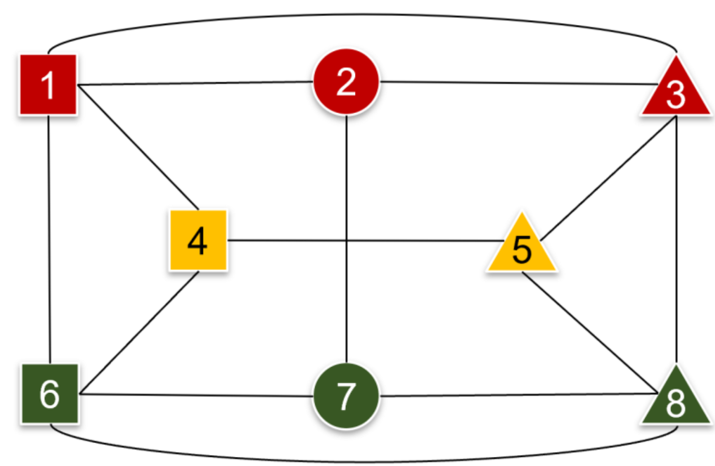
Compatible Class（相容类）

- Given a compatible relation $\boldsymbol{A}$, $\boldsymbol{C}$ is called **compatible class** if:
  - $\boldsymbol{C} \subseteq \boldsymbol{A}$
  - $(\boldsymbol{x} \in \boldsymbol{A}) \wedge (\forall \boldsymbol{y})((\boldsymbol{y} \in \boldsymbol{C}) \rightarrow \boldsymbol{xRy})$
- Examples:
  - $\{1,4,6\}$
  - $\{1,4\}$
  - $\{3,8\}$
  - …



Maximal Compatible Class(极大相容类)

- A compatible class C is called **maximal compatible class** if C is not the subset of any compatible class, donate by C_R.
- Examples:
  - $\{1,4,6\}$
  - $\{1,2,3\}$
  - $\{2,7\}$
  - $\{3,5,8\}$
  - …

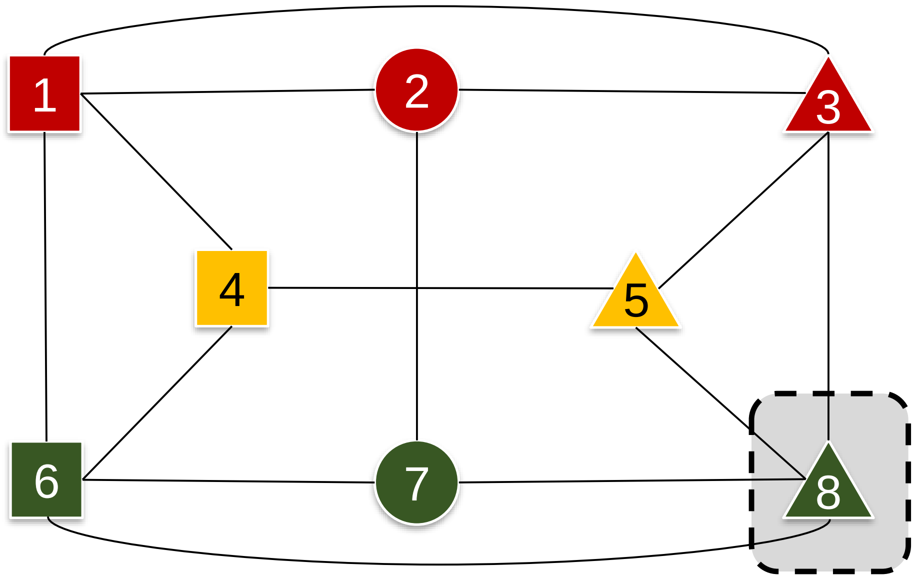


Maximal Compatible Class

For any compatible class C on set $\boldsymbol{A}$, there exists a maximal compatible class C_R such that $C \subseteq C_R$

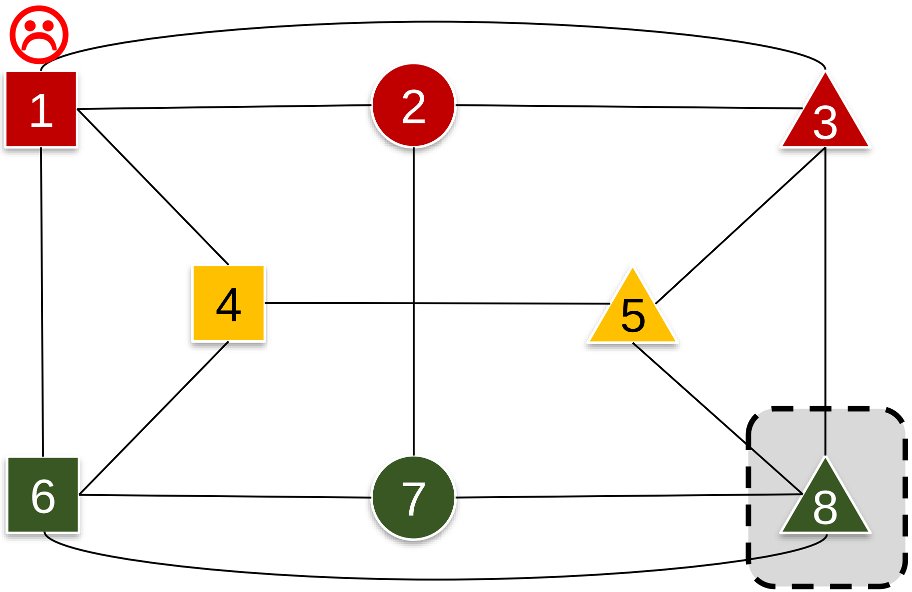
- We can generate a sequence such that $\boldsymbol{C_0} \subseteq \boldsymbol{C_1} \subseteq \boldsymbol{C_2} \dots \subseteq \boldsymbol{C_R}$
  - $\boldsymbol{C_0 = C}$
  - $\boldsymbol{C_{i+1} = C_i \cup \{a_j\}}$, where j is the minimal index such that $\boldsymbol{a_j \notin C_i \wedge (x \in C_i \to a_j R x)}$

Maximal Compatible Class



$$C_0 = C = \{8\}$$

Maximal Compatible Class



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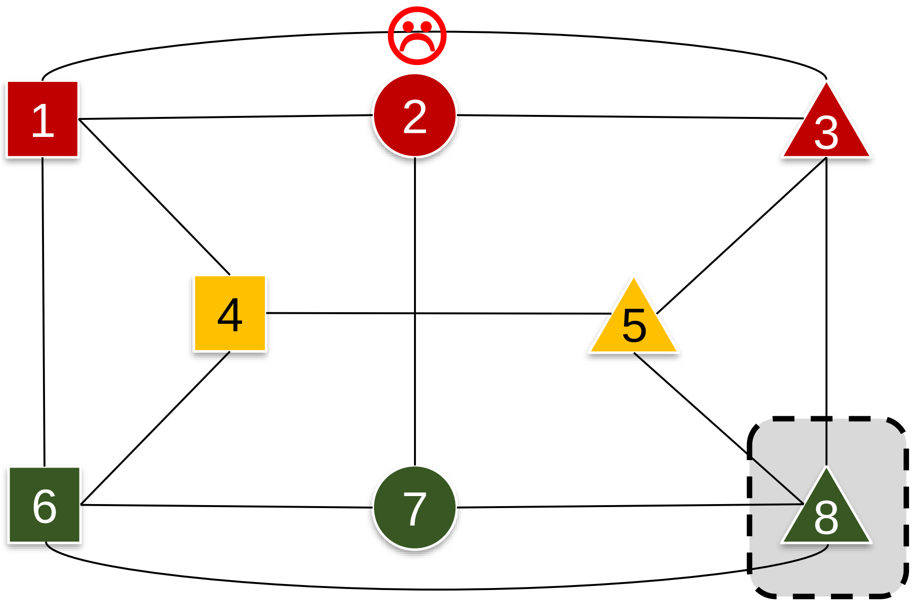

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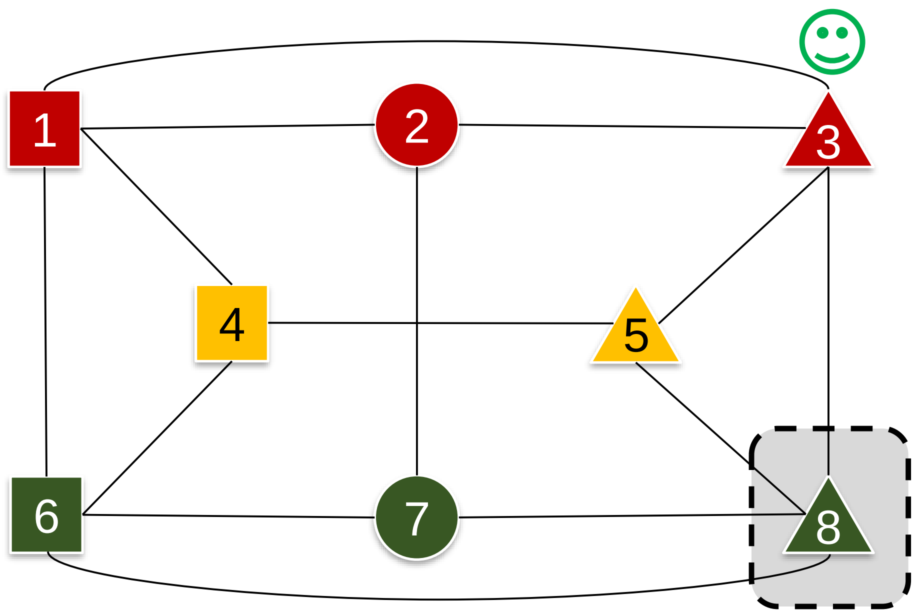

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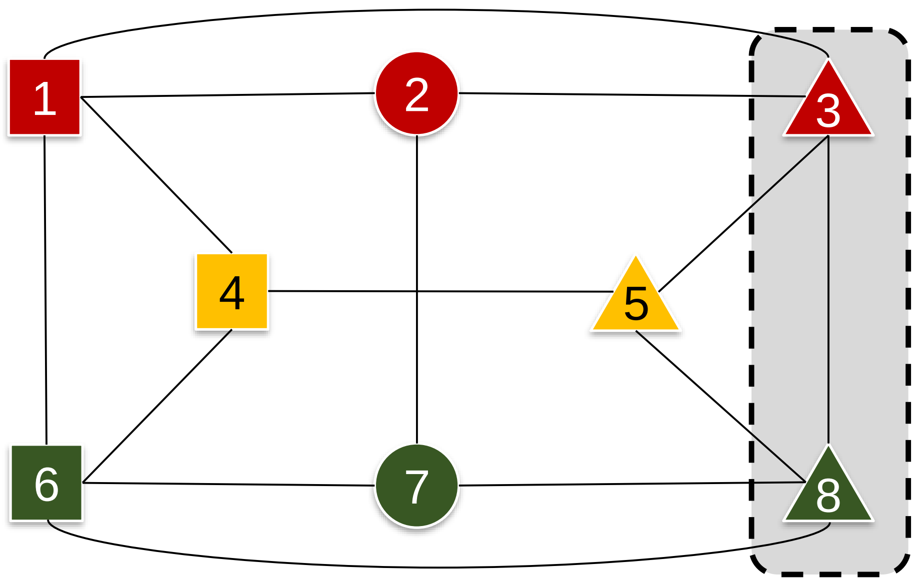
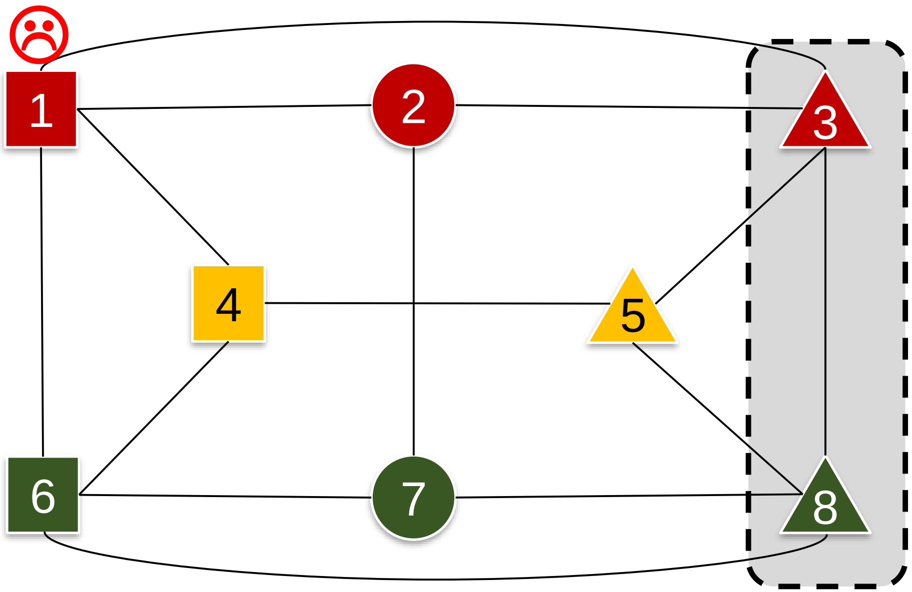


$$C_0 = C = \{8\}$$

$$C_1 = \{3,8\}$$

Maximal Compatible Class



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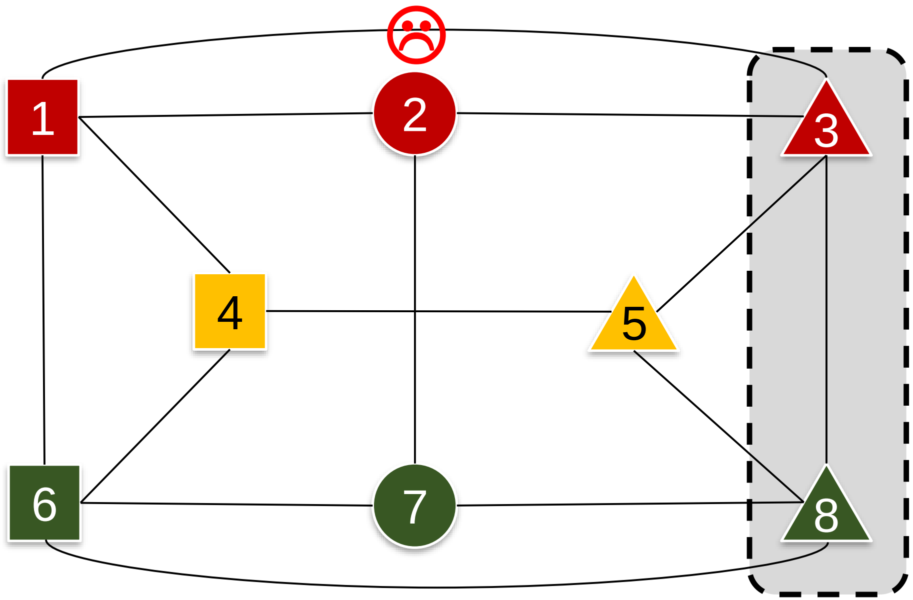

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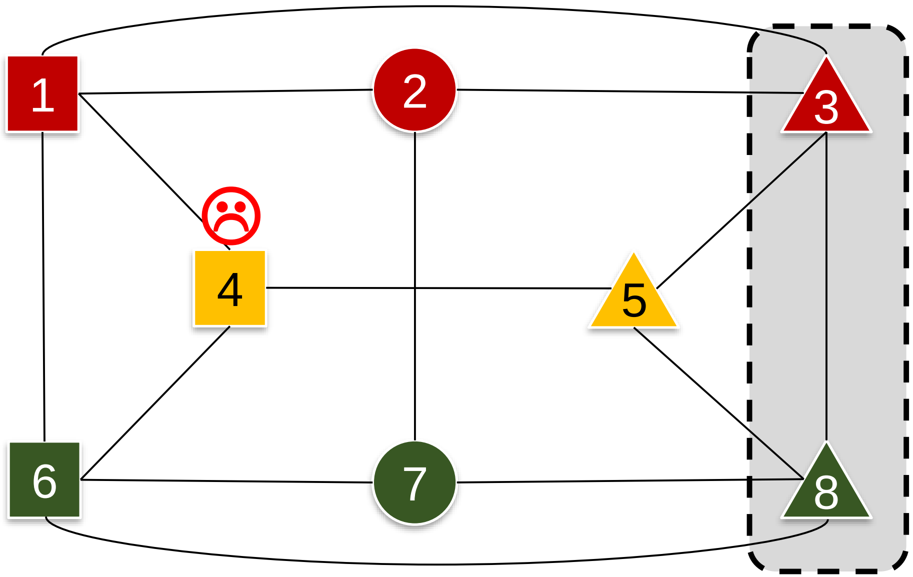

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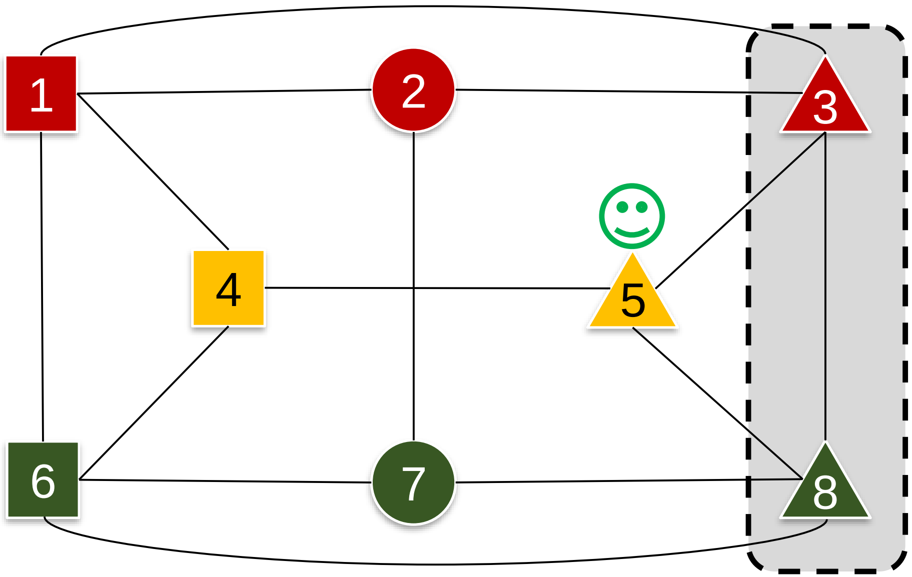

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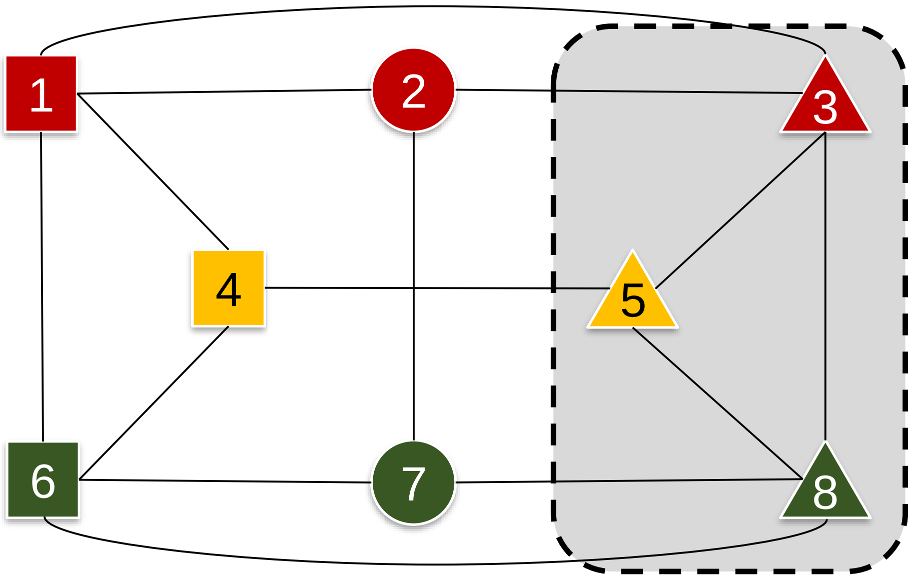
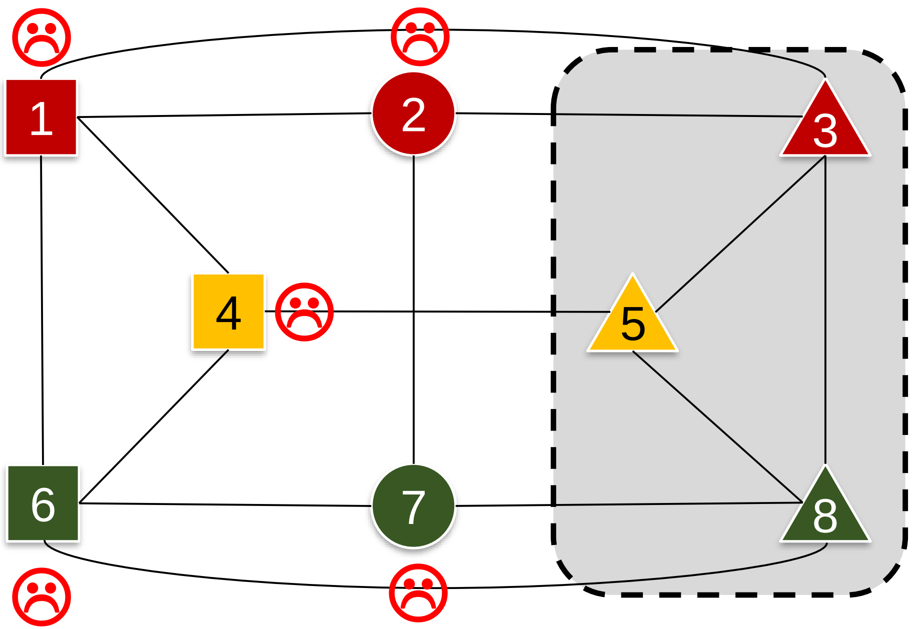


$$C_0 = C = \{8\}$$

$$C_1 = \{3,8\}$$

$$C_2 = \{3,5,8\}$$

Maximal Compatible Class



$$C_0 = C = \{8\}$$

$$C_1 = \{3,8\}$$

$$C_2 = \{3,5,8\}$$

$$\boldsymbol{C_R = C_2 = \{3, 5, 8\}}$$

Compatible Relation and Covers

- Given a compatible relation $\boldsymbol{R}$ on a set $\boldsymbol{A}$, the set of all maximal compatible classes is a cover of $\boldsymbol{A}$, called the complete cover (完全覆盖) of $\boldsymbol{A}$, donate by $\boldsymbol{C_R(A)}$.
  - $C_R(A)$ is unique.
- A compatible relation of $\boldsymbol{A}$ can generate a complete cover of $\boldsymbol{A}$
- A cover of $\boldsymbol{A}$ can generate a compatible relation of $\boldsymbol{A}$
- Two different covers can generate the same compatible relation.

Next

Capture the structure of a problem ⟨continued⟩

- Partial Order, total order.
- Closures

Other fundamental tools.

- Functions