Discrete Mathematics
(for Computer Science)

Zhaoguo Wang, Zeyu Mi

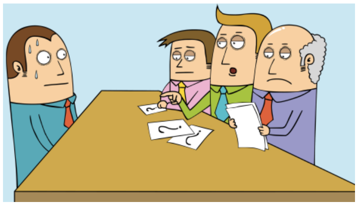
IPADS

Why Choose CDM?

4 yrs in SJTU

For graduation

Graduate school



Interview

Startup

Programmer
/ Researcher

Why Choose CDM?

v.s.

GPT-4

What makes you distinguished?

Why Choose CDM?

V.S.

GPT-4

What makes you distinguished?

Understand "laws of physics" in CS

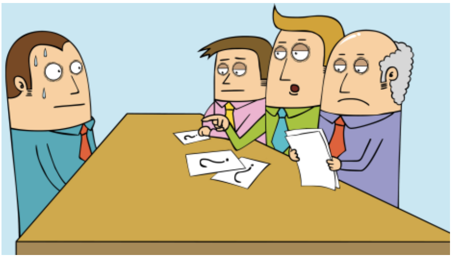
Be able to answer the fundamental questions.

Questions To Be Answered In CDM

What problems can you solve with a computer?

How can you be certain of your answer to these questions?

How Do These Two Questions Affect Each Path?



Interview

Interview Question:

what languages r u good at?

Java, Java Spring, Ruby, Python

Java, C++, .Net, C, Go, Rust

Besides above all, assembly language, Coq, TLA and so on…

How Do These Two Questions Affect Each Path?

Graduate school

Major concern by Profs.

Is the student a logical person?

He got a grade of A for CDM.

He can have logical thinking and answer the questions in a very logical way.

How Do These Two Questions Affect Each Path?

Startup

When talk to investors

The first feeling is important

His story makes sense.

He has a very clear mind.

How Do These Two Questions Affect Each Path?

Programmer / Researcher

How do you know your program is correct?

There is no segmentation fault.

It passes all tests.

I can prove it.

大纲

- 图论
  - 图的基本概念
  - 道路与回路
  - 树
- 逻辑
  - 命题逻辑
  - 谓词逻辑
  - 公理系统
- 集合论
  - 集合
  - 关系
  - 函数

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?

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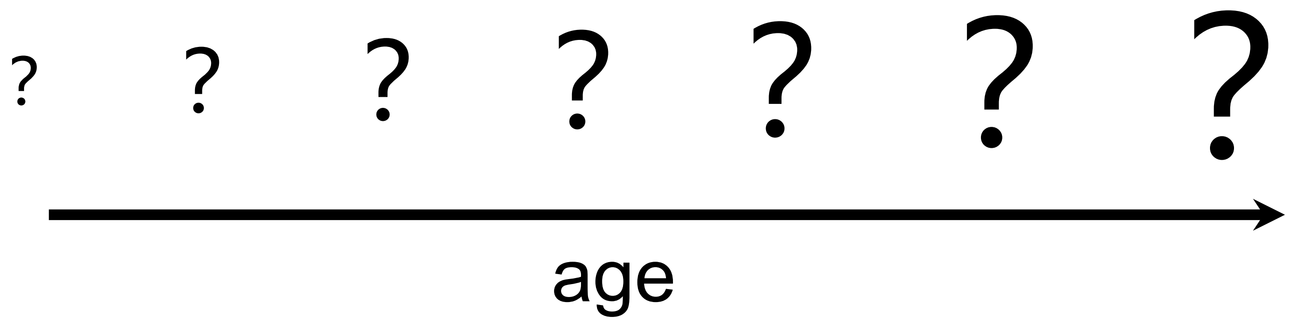

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  - 集合
  - 关系
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这将不是一门普通的离散数学

? ? ? ? ? ? ?

age

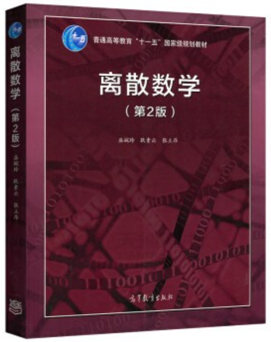
Be Practical than Attractive

Is it solvable?

```
bool is_power2(unsigned int v)
{
    unsigned int i;
    for(i = 0; i <= 31; ++i)
    {
      if(v == (1 << i))
          return true;
    }
    return false;
}
```

How to prove its correctness?

1983



2015

离散数学 =

- 图论
- 数理逻辑
- 集合论
- 组合数学
- ……

What we teach:

Discrete Mathematics
for **Computer Science**

NYU

RICE
Unconventional Wisdom

- Automata
- Turing machine
- Computation theory

- Satisfiability problem
- Satisfiability modulo theory

How to module a program

How to prove a program

Three Parts of CDM

Part I. Logic Reasoning. (*~10 Weeks*)

How to prove the correctness?

Part II. Computability. (*~ 5 Weeks*)

Is the problem computable (solvable)?

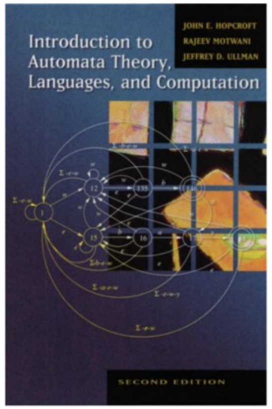
Part III. Applications. (*~ 1 Weeks*)

Solve the problem in real systems.

References

Textbooks

《数理逻辑与集合论》第2版
- 石纯一 著，清华大学出版社



John E. Hopcroft, Rajeev Motwani and Jeffrey D. Ullman, Introduction to Automata Theory, Languages, and Computation, Pearson, 2001

To Be Distinguished, You Need To

Take

- ✓ Lectures
- ✓ Recitation (Optional)

Do

- ✓ 3 Labs
- ✓ Homework (Preview + Review)

Pass

- ✓ Quiz
- ✓ Final exam

Grading Breakdown

- Lecture + Homework + Quiz + ~~(Optional Lab)~~ 35%
- Lab 25%
- **Final 40%**

Policies

You must work alone on all assignments

- You may post questions on Canvas.
- You are encouraged to answer others' questions, but refrain from explicitly giving away solutions.

Labs & Exercises

- Preview assignments due at course starting time (10:00am on Mon. & 14:00am on Thu.) on the due date
- Review assignments due at 11:59pm on the due date
- Everybody has 5 grace days
    - Must make the claim before the due
- Zero score after the due

Integrity and Collaboration Policy

We will enforce the policy strictly.

1. The work that you turn in must be yours
2. You must acknowledge your influences
3. You must not look at, or use, solutions from prior years or the Web, or seek assistance from the Internet
4. You must take reasonable steps to protect your work
   - You must not publish your solutions
5. If there are inexplicable discrepancies between exam and lab performance, we will over-weight the exam and interview you.

Faculty Information

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Three Parts of CDM

Part I. Logic Reasoning. (*~10 Weeks*)

How to prove the correctness?

Part II. Computability. (~ *5 Weeks*)

Is the problem computable (solvable)?

Part III. Applications. (~ *1 Weeks*)

Solve the problem in real systems.

Class Info

Website:
http://ipads.se.sjtu.edu.cn/courses/cdm

Canvas:
https://oc.sjtu.edu.cn/courses/81416

What is Discrete Mathematics?

What is the most formal definition of Discrete Mathematics?

Formal Definition

Discrete Mathematics is the branch of mathematics concerned with the study of mathematical structures that are fundamentally **countable** or **distinct** and **separable**. Unlike continuous mathematics, which deals with objects that can vary smoothly (such as real numbers and calculus), discrete mathematics deals with objects that can assume only distinct, separated values.

Its primary focus is on:

1. **Finite** or **countably infinite** sets.
2. Structures that emphasize **relationships** and **connectivity** between distinct elements.
3. Processes that involve **finite-state** changes and **step-by-step** procedures (algorithms).

The field is characterized by its emphasis on **precision**, **rigorous proof**, and **methodical reasoning** about discrete objects.

Example I – Number Theory

Definition: Number theory is a branch of pure mathematics devoted primarily to the study of **the integers and arithmetic functions**.

1. **Finite** or **countably infinite** sets.

1, 2, 3, 4, 5, …

2. Structures that emphasize **relationships and connectivity** between distinct elements**.**

2 > 1, 4 > 5

3. Processes that involve **finite-state changes and step-by-step procedures** (algorithms).

1 + 1 = 2

2 + 1 = 3

Example II – Set Theory

Definition: Its core subject of study is sets — that is, the **totality of objects with specific properties or satisfying certain conditions**.

1. **Finite** or **countably infinite** sets.

   $\{2\}, \{2, 4\}, \{2, 4, 6, 8\}, \ldots$

2. Structures that emphasize **relationships and connectivity** between distinct elements.

   $\{2\} \subseteq \{2,4\}$

3. Processes that involve **finite-state changes and step-by-step procedures** (algorithms).

   $\{2\} \cup \{4\} = \{2,4\}$

Example III – Logic Reasoning

Definition: Its provides the formal system for manipulating **discrete truth values**, establishing the validity of **relationships between discrete statements.**

1. **Finite** or **countably infinite** sets.

True, False, Statements …

2. Structures that emphasize **relationships and connectivity** between distinct elements.

"Discrete Math is Math" **= True**

3. Processes that involve **finite-state changes and step-by-step procedures** (algorithms).

"Discrete Math is Math" ∧ "I Teach Discrete Math" → "I Teach Math"

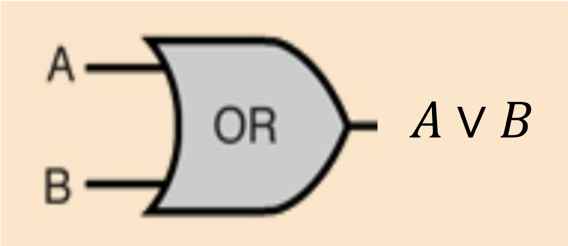
Three Parts of CDM

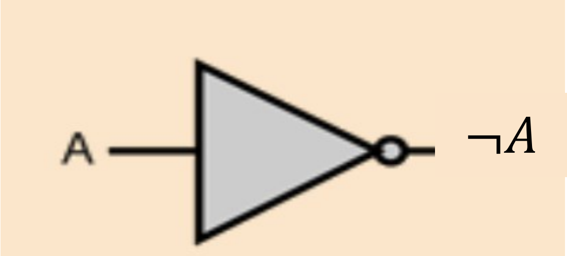
Part I. Logic Reasoning.

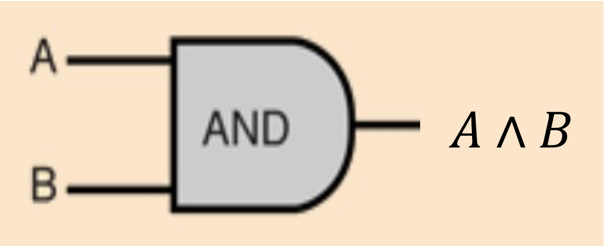
Part II. Computability.

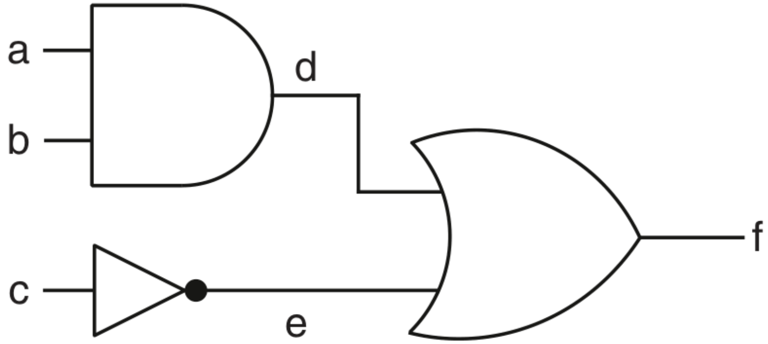
Part III. Applications.

Part I: Logic Reasoning – Integrated Circuits









Use Logic Reasoning to

- Check the logic of a designed circuit
- Verify if an optimized circuit is equivalent to the original one

Part I: Logic Reasoning

What is Logic Reasoning used for in LLM era?

Improve and ensure the certainty and correctness of the answer!

Part I: Logic Reasoning

How does Logic Reasoning improve the certainty?

Input:

```
A cage has 35 heads and 94 legs. Chickens have 1 head and 2 legs. Rabbits have 1 head
and 4 legs. How many chickens and rabbits are there? **Let's think step by step.**
```

- **Output:** `Chickens: 23, Rabbits: 12` (The model might get this right, but it's a guess. It could just as easily make a calculation error with no transparent process, leaving the user unsure of the answer's validity.) [3]

Part I: Logic Reasoning

How does Logic Reasoning improve the certainty?

Answer: Chain-of-Thought (COT 思维链)

- **Input:**

```
A cage has 35 heads and 94 legs. Chickens have 1 head and 2 legs. Rabbits have 1 head and 4 legs. How many chickens and rabbits are there? **Let's think step by step.**
```

- **Output:**

`> **Step 1: Define variables.**

> Let c be the number of chickens.
> Let r be the number of rabbits.` 5

`> **Step 2: Translate the problem into equations based on heads and legs.**

> From heads: $c + r = 35$ (since each animal has one head).
> From legs: $2c + 4r = 94$ (since chickens have 2 legs each and rabbits have 4 legs each).` 4 7

`> **Step 3: Solve the system of equations.**

> First, simplify the leg equation by dividing all terms by 2: $c + 2r = 47$.
> Now we have:
> $c + r = 35$ ...(equation 1)
> $c + 2r = 47$ ...(equation 2)
> Subtract equation 1 from equation 2: $(c + 2r) - (c + r) = 47 - 35$, which simplifies to $r = 12$.
> Substitute $r = 12$ back into equation 1: $c + 12 = 35$, so $c = 23$.` 5 7

`> **Step 4: Verify the solution.**

> Check heads: $23 + 12 = 35$ ✓
> Check legs: $(23 \times 2) + (12 \times 4) = 46 + 48 = 94$ ✓` 6

`**Therefore, there are 23 chickens and 12 rabbits.**`

Part I: Logic Reasoning

Google DeepMind

How does Logic Reasoning ensure the correctness?

SCIENCE

AlphaEvolve: A Gemini-powered coding agent for designing advanced algorithms

14 MAY 2025

By AlphaEvolve team

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Competitive programming with AlphaCode

8 DECEMBER 2022

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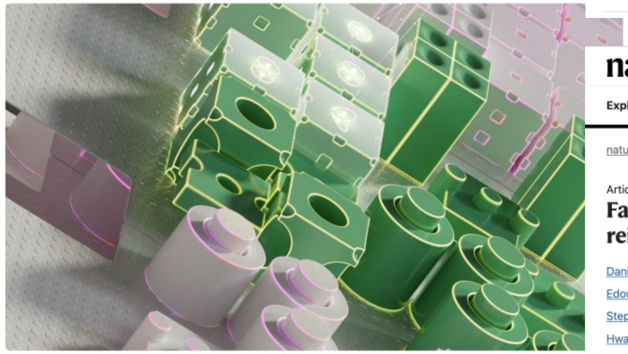

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RESEARCH ARTICLE | COMPUTER SCIENCE

Competition-level code generation with AlphaCode

YUJIA LI, DAVID CHOI, JUNYOUNG CHUNG, NATE KUSHMAN, JULIAN SCHRITTWIESER, RÉMI LEBLOND, TOM ECCLES, JAMES KEELING, FELIX GIMENO, [...], AND ORIOL VINYALS +16 authors Authors Info & Affiliations

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Faster sorting algorithms discovered using deep reinforcement learning

Daniel J. Mankowitz, Andrea Michi, Anton Zhernov, Marco Gelmi, Marco Selvi, Cosmin Paduraru, Edouard Leurent, Shariq Iqbal, Jean-Baptiste Lespiau, Alex Ahern, Thomas Köppe, Kevin Millikin, Stephen Gaffney, Sophie Elster, Jackson Broshear, Chris Gamble, Kieran Milan, Robert Tung, Minjae Hwang, Taylan Cemgil, Mohammadamin Barekatain, Yujia Li, Amol Mandhane, Thomas Hubert, ... David Silver + Show authors

Part I: Logic Reasoning

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Nanjing University Xianlin Campus.

软件工程技术研究漫谈

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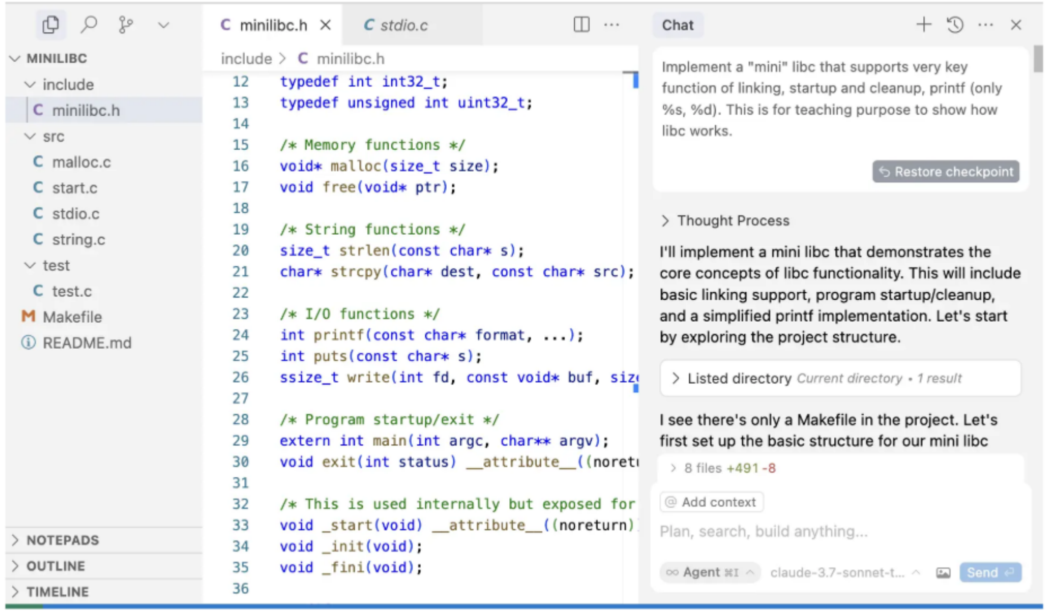
如何 “实现” 一个 libc?



(当然是让 “老师” 当着你的面实现一个！)

然后打脸 😂

我修了好多 bug

- Inline assembly 忘记 %%
- _start 的 rsp 已经被改过了
- mmap 的参数顺序错了
- munmap 忘记处理头上的 8 字节 size
  - 也只是让 test.c 能正常运行了
  - 不知道还有没有更多的问题

代码：谨慎使用 AIGC

Part I: Logic Reasoning

How does Logic Reasoning ensure the correctness?

Answer: Formal Verification (LLM + FM)

SCIENCE

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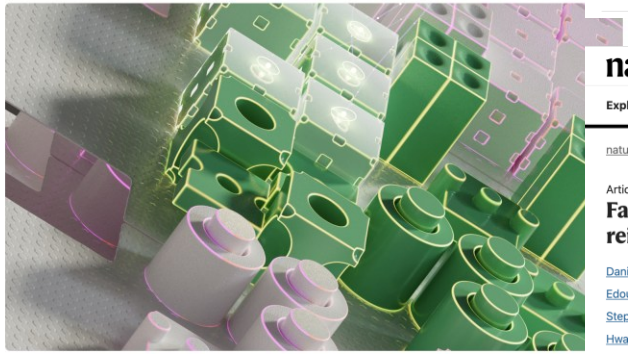

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Part I: Logic Reasoning

deepseek

Write a C function to determine whether an integer is a power of 2?

c

Copy Download

```c
#include <stdbool.h>

bool isPowerOfTwo(unsigned int n) {
    if (n == 0) {
        return false;
    }
    int count = 0;
    while (n) {
        count += n & 1; // Check if the least significant bit is 1
        n >>= 1;        // Shift bits to the right
    }
    // If only one bit was set, it's a power of two.
    return (count == 1);
}
```

Part I: Logic Reasoning

Can you simplify the generated C code?

Here's a simple C function to check if an integer is a power of 2:

c   Copy   Download

```c
#include <stdbool.h>

bool isPowerOfTwo(int n) {
    if (n <= 0) {
        return false;
    }
    return (n & (n - 1)) == 0;
}
```

Part I: Logic Reasoning

deepseek

can you further simplify it?

Here's the simplest C function to determine if an integer is a power of 2:

c Copy Download

```c
#include <stdbool.h>

bool isPowerOfTwo(int n) {
    return n > 0 && (n & (n - 1)) == 0;
}
```

Part I: Logic Reasoning

How could you ensure the most simplified version is correct?

1. Test with Key Values

c Copy Download

```c
#include <stdio.h>

int isPowerOfTwo(int n) { return n > 0 && !(n & (n - 1)); }

int main() {
    // Powers of 2 (should return 1)
    printf("1: %d\n", isPowerOfTwo(1));     // 2^0
    printf("2: %d\n", isPowerOfTwo(2));     // 2^1
    printf("4: %d\n", isPowerOfTwo(4));     // 2^2
    printf("8: %d\n", isPowerOfTwo(8));     // 2^3
    printf("16: %d\n", isPowerOfTwo(16));   // 2^4
    printf("1024: %d\n", isPowerOfTwo(1024)); // 2^10

    // Non-powers of 2 (should return 0)
    printf("0: %d\n", isPowerOfTwo(0));     // edge case
    printf("3: %d\n", isPowerOfTwo(3));
    printf("6: %d\n", isPowerOfTwo(6));
    printf("10: %d\n", isPowerOfTwo(10));
    printf("15: %d\n", isPowerOfTwo(15));

    // Negative numbers (should return 0)
    printf("-1: %d\n", isPowerOfTwo(-1));
    printf("-4: %d\n", isPowerOfTwo(-4));
    printf("-8: %d\n", isPowerOfTwo(-8));

    return 0;
}
```

2. Mathematical Verification

Let's trace through the logic:

- For n = 8 (1000 in binary): `8 > 0 && !(8 & 7)` → `true && !(1000 & 0111)` → `true && !(0000)` → `true && true` → **1** ✓
- For n = 6 (0110 in binary): `6 > 0 && !(6 & 5)` → `true && !(0110 & 0101)` → `true && !(0100)` → `true && false` → **0** ✓
- For n = 0: `0 > 0 && ...` → `false && ...` → **0** ✓
- For n = -4: `-4 > 0 && ...` → `false && ...` → **0** ✓

3. Property Verification

Powers of 2 have exactly one '1' bit in their binary representation:

- 1: 0001 ✓
- 2: 0010 ✓
- 4: 0100 ✓
- 8: 1000 ✓

The expression `n & (n - 1)` clears the least significant '1' bit. If only one '1' bit exists (power of 2), this results in 0.

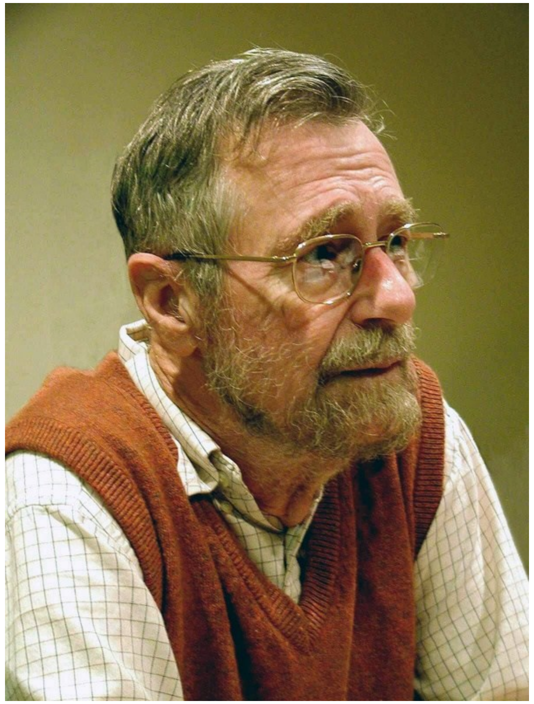
Part I: Logic Reasoning

how could you ensure your test cases are correct and cover all branches?

2. Comprehensive Test Suite

```c
#include <stdio.h>
#include <limits.h>

int isPowerOfTwo(int n) { return n > 0 && !(n & (n - 1)); }

void test_case(int n, int expected, const char *description) {
    int result = isPowerOfTwo(n);
    printf("%s: %d -> %d (expected %d) %s\n",
           description, n, result, expected,
           result == expected ? "✓" : "✗");
}

int main() {
    printf("=== Testing Power of 2 Function ===\n");

    // Branch 1: n > 0 → false (n ≤ 0)
    test_case(0, 0, "Zero");
    test_case(-1, 0, "Negative one");
    test_case(-4, 0, "Negative four");
    test_case(-8, 0, "Negative eight");
    test_case(INT_MIN, 0, "Minimum integer");
    test_case(3, 0, "Three (not power of 2)");
    test_case(5, 0, "Five (not power of 2)");
    test_case(6, 0, "Six (not power of 2)");
    test_case(7, 0, "Seven (not power of 2)");
    test_case(9, 0, "Nine (not power of 2)");
    test_case(10, 0, "Ten (not power of 2)");
    test_case(15, 0, "Fifteen (not power of 2)");
    test_case(1023, 0, "1023 (not power of 2)");
    test_case(INT_MAX, 0, "Maximum integer (not power of 2)");

    // Branch 4: !(n & (n-1)) → true (power of 2)
    test_case(1, 1, "One (2^0)");
    test_case(2, 1, "Two (2^1)");
    test_case(4, 1, "Four (2^2)");
    test_case(8, 1, "Eight (2^3)");
    test_case(16, 1, "Sixteen (2^4)");
    test_case(32, 1, "Thirty-two (2^5)");
    test_case(64, 1, "Sixty-four (2^6)");
    test_case(128, 1, "One twenty-eight (2^7)");
    test_case(256, 1, "Two fifty-six (2^8)");
    test_case(512, 1, "Five twelve (2^9)");
    test_case(1024, 1, "Ten twenty-four (2^10)");
    test_case(2048, 1, "Two forty-eight (2^11)");
    test_case(4096, 1, "Four ninety-six (2^12)");

    // Edge cases around powers of 2
    test_case(1023, 0, "1024-1");
    test_case(1025, 0, "1024+1");
    test_case(2047, 0, "2048-1");
    test_case(2049, 0, "2048+1");
```

Part I: Logic Reasoning



Edsger Wybe Dijkstra

Testing shows the presence not
the absence of bugs.

Part I: Logic Reasoning

Could you prove the correctness of the code?

Theorem:

The function `isPowerOfTwo(n)` returns `true` **if and only if** `n` is a positive integer and $n = 2^k$ for some integer `k ≥ 0`.

Proof:

1. (⇒) If `isPowerOfTwo(n)` returns `true`, then `n` is a power of two.

- By the function's definition, `n > 0` and `(n & (n - 1)) = 0`.
- The condition `(n & (n - 1)) = 0` implies that `n` has a single `1` bit in its binary representation. This is because subtracting `1` flips the least significant `1` bit and all subsequent `0`s, and the bitwise AND is zero only if no higher bits remain.
- A positive integer with exactly one `1` bit is, by definition, a power of two. ■

2. (⇐) If `n` is a positive power of two, then `isPowerOfTwo(n)` returns `true`.

- Let $n = 2^k$ for some `k ≥ 0`. Then `n > 0`.
- The binary representation of `n` is a `1` followed by `k` `0`s.
- The binary representation of `n - 1` is `k` `1`s.
- Therefore, `n & (n - 1) = 0` (the `1` in `n` aligns with a `0` in `n-1`, and all other bits are `0` in `n`).
- Thus, both conditions `n > 0` and `(n & (n - 1)) = 0` are satisfied. ■

By proving both directions, we conclude the function is correct.

Part I: Logic Reasoning

deepseek

How could you ensure your proof is correct?

1. Verify the Logical Structure

The proof has a standard and robust structure: an **if-and-only-if** statement proven in two parts (⇒ and ⇐). This structure ensures no gaps in logic.

- **Part 1 (Soundness):** Proves the function has no **false positives**. If it returns `true`, the input *must be* a power of two.
- **Part 2 (Completeness):** Proves the function has no **false negatives**. If the input *is* a power of two, the function *will* return `true`.

This two-part approach is a recognized best practice for validating equivalence.

2. Validate the Core Insight

The entire proof hinges on one key insight: **For `n > 0`, `n & (n - 1) == 0` is equivalent to `n` having a single `1` in its binary representation.**

- **Justification:** This is a well-known and proven property in bitwise arithmetic. It can be found in computer science literature and textbooks (e.g., *Hacker's Delight* by Henry S. Warren, Jr.). Relying on such an established lemma strengthens the proof's credibility.
- **Sanity Check with Examples:**
  - `n = 8` (1000). `n-1 = 7` (0111). `1000 & 0111 = 0000`. ✓
  - `n = 6` (0110). `n-1 = 5` (0101). `0110 & 0101 = 0100 != 0`. ✓
  - `n = 0`. `n-1 = -1`. `0 & (-1) = 0`, but `n > 0` is false, so the function correctly returns `false`. This validates the necessity of the `n > 0` check.

Testing the core logic with concrete examples confirms the abstract reasoning.

3. Check for Edge Cases

A correct proof must account for all possible inputs, especially edge cases.

- **n = 0:** Addressed. `0` is not positive, so the function returns `false`. The proof explicitly states `n > 0`.
- **n = 1 (2^0):** `1 > 0` is true. `1 & 0 = 0` is true. The function returns `true`. The proof holds for the smallest power of two.
- **Negative Numbers:** The function's first condition `n > 0` filters them out correctly. The proof only concerns itself with positive integers.
- **Large Numbers:** The proof is based on the properties of binary representation, which hold for all integers within the language's limits (e.g., 32-bit or 64-bit `int`). The logic is scale-invariant.

4. Peer Review and Formalization

- **Explain it to someone else (or a rubber duck):** Articulating the proof's steps forces you to clarify your reasoning and exposes hidden assumptions or leaps in logic. The proof presented is clear and easy to follow.
- **Formalize the bit-level argument:** The proof can be made more formal by denoting the position of the least significant `1` bit.
  - Let `n` be a positive integer. Let `i` be the position of its least significant `1` bit. Then `n` can be represented as: `n = ...a_{i+1}a_i 00...0` where `a_i = 1`.
  - Then `n-1 = ...a_{i+1}0 11...1`.
  - Therefore, `n & (n-1) = ...a_{i+1}0 00...0`.
  - For `n & (n-1)` to be `0`, there must be no `1` bits to the left of position `i` (i.e., all `a_{i+1}...` must be `0`). This means the `1` at position `i` was the only `1` in the number.

5. Contrast with Incorrect Alternatives

A good way to ensure a proof is correct is to see why common *incorrect* solutions fail and confirm your proof accounts for this.

- **Incorrect alternative:** `return (n & (n - 1)) == 0;`
  - **Why it's wrong:** Fails for `n=0`. `0 & (-1) = 0`, so it would return `true` for `0`, which is not a power of two.
  - **Does our proof cover this?** Yes. The proof explicitly requires `n > 0` as a necessary condition in both directions.

More difficult to ensure the correctness

Part I: Logic Reasoning

deepseek

How to ensure the proof of your proof is correct?

1. The Social Process: Peer Review

This is the most common and practical method in both academia and industry.

- **How it works:** You present your proof to other experts. They critique the logic, identify hidden assumptions, challenge steps, and attempt to find counterexamples.
- **Why it works:** It leverages the collective intelligence and skepticism of a community. Errors that one person overlooks are often caught by another.
- **Analogy:** It's like having multiple senior engineers review a piece of critical code.
- **For our power-of-two proof:** Explaining the two-part if-and-only-if structure and the bit-level argument to another programmer or mathematician is a form of peer review. If they agree it's sound, your confidence increases.

2. The Mechanical Process: Exhaustive Testing (for finite domains)

While not a proof itself, testing can provide extreme confidence for functions with limited input spaces.

- **How it works:** For a 32-bit integer, there are only 32 positive powers of two (2^0 to 2^{31}). You can write a test harness that iterates over all ~4.3 billion possible `int` values and verifies that the function returns `true` **only** for those 32 values and `false` for all others.
- **Why it works:** It provides a complete, empirical verification for the given domain (e.g., 32-bit integers). It can catch off-by-one errors and missed edge cases that a logical proof might have overlooked.
- **Limitation:** It only works for finite domains. It doesn't constitute a general proof for all possible integers (e.g., in arbitrary-precision arithmetic), but for a C `int`, it's effectively perfect.

3. The Formal Process: Constructing a Formal Proof

This is the gold standard. You break down the proof into such small, precise steps that validity can be verified automatically.

Even Harder

- **Step 1: Formal Specification:** Precisely define the function's behavior in a formal l our function, we would specify:
  - `isPowerOfTwo(n) ⇔ (n > 0) ∧ (∃k : Nat, n = 2^k)`
- **Step 2: Use a Proof Assistant:** Use a tool like **Coq**, **Lean**, **Isabelle/HOL**, or **ACL2**. These are not just programming languages but systems for writing machine-checkable proofs.
  - You would **define** the function in the tool's language.
  - You would **state the theorem** you want to prove (`∀ n : int, isPowerOfTwo n = true ↔ n > 0 ∧ ∃ k, n = 2^k`).
  - You would **interactively construct the proof** within the tool. The tool breaks the goal into smaller subgoals. For each step (e.g., "rewrite using the definition of `&`", "case analysis on `n` "), the tool ensures the logical rule is applied correctly.
- **Why it works:** The proof assistant acts as an infallible, ultra-pedantic peer reviewer. It has a built-in kernel of logical rules and will not a justified. It eliminates human error in the va
- **What it ensures:** The proof of your proof is the axioms to the final theorem, has been assumptions are the correctness of the pr

4. The Foundational Process: Minimizing Trust

Even with a proof assistant, you must trust its core logic ("kernel"). To ensure even *that* is correct, the field of **verified computing** takes it a step further.

- **How it works:** The source code of the proof assistant's kernel itself is mathematically proven to be correct. This is called a **verified kernel**.
- **Example:** The **CompCert** C compiler is a famous project where the compiler itself is formally verified in Coq to ensure it translates source code correctly. You could, in theory, verify a proof assistant's kernel with another, more minimal, proof assistant.
- **This is the final step.** It minimizes the "trust base" to a very small set of axioms and the physical laws governing the computer hardware.

Part I: Computer Aided Reasoning

Prompt: Code Requirement

Write a C function to determine whether an integer is a power of 2?

deepseek

Computer Aided Reasoning (Automatic Verifier)

Part I: Computer Aided Reasoning

```
bool xor(unsigned int v)
{
    return v ^ v;
}
```

$$
\begin{aligned}
&(((\neg v1) \wedge v1) \vee (v1 \wedge (\neg v1)))\vee \\
&(((\neg v2) \wedge v2) \vee (v2 \wedge (\neg v2)))\vee \\
&(((\neg v3) \wedge v3) \vee (v3 \wedge (\neg v3)))\vee \\
&\ldots \\
&(((\neg v32) \wedge v32) \vee (v32 \wedge (\neg v32)))
\end{aligned}
$$

A Propositional Formula

1.1 Propositional Logic

Step 1. Convert it into mathematical formula.

Step 2. Ask the computer to solve the formula.

Part I: Computer Aided Reasoning

```
void swap(int a, int b)
{
     a ^= b;
     b ^= a;
     a ^= b;
}
```

$$(a1' \leftrightarrow (((((a1 \wedge (\neg b1)) \vee ((\neg a1) \wedge b1)))\wedge$$
$$(\neg((((((a1 \wedge (\neg b1)) \vee ((\neg a1) \wedge b1))) \wedge (\neg b1))\vee$$
$$((\neg(((a1 \wedge (\neg b1)) \vee ((\neg a1) \wedge b1)))) \wedge b1)))))\vee$$
$$((\neg(((a1 \wedge (\neg b1)) \vee ((\neg a1) \wedge b1))))\wedge$$
$$((((((a1 \wedge (\neg b1)) \vee ((\neg a1) \wedge b1))) \wedge (\neg b1))\vee$$
$$((\neg(((a1 \wedge (\neg b1)) \vee ((\neg a1) \wedge b1)))) \wedge b1))))))\wedge$$
$$(b1' \leftrightarrow (((((a1 \wedge (\neg b1)) \vee ((\neg a1) \wedge b1))) \wedge (\neg b1))\vee$$
$$((\neg(((a1 \wedge (\neg b1)) \vee ((\neg a1) \wedge b1)))) \wedge b1)))\wedge$$

...

A Propositional Formula

1.1 Propositional Logic

Step 1. Convert it into mathematical formula.

Step 2. Ask the computer to solve the formula.

It is complicated for both human being and computer.

Part I: Computer Aided Reasoning

```
void swap(int a, int b)
{
    a ^= b;
    b ^= a;
    a ^= b;
}
```

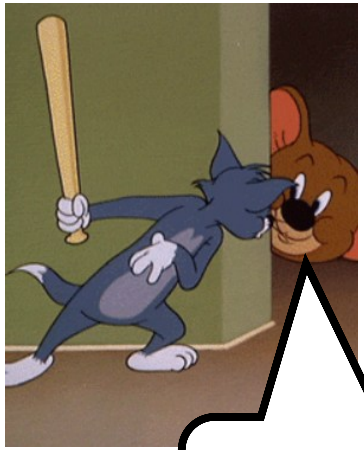
$$(a' = XOR(XOR(a,b), XOR(b, XOR(a,b))))\wedge$$
$$(b' = XOR(b, XOR(a,b)))$$

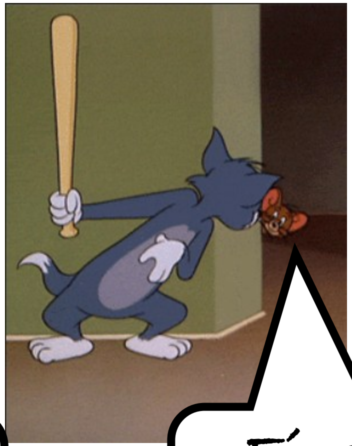
A Predicate Formula

1.1 Propositional Logic

Step 1. Convert it into mathematical formula.

Step 2. Ask the computer to solve the formula.





Part I: Computer Aided Reasoning

```
void swap(int a, int b)
{
    a ^= b;
    b ^= a;
    a ^= b;
}
```

$$(a' = XOR(XOR(a,b), XOR(b, XOR(a,b))))\wedge$$
$$(b' = XOR(b, XOR(a,b)))$$

A Predicate Formula

1.1 Propositional Logic

Step 1. Convert it into mathematical formula.

Step 2. Ask the computer to solve the formula.

1.2 First Order Logic

Step 1. Convert it into first logic formula.

Step 2. Ask the computer to solve the formula.

Part I: Computer Aided Reasoning

```
int clone(unsigned int n)
{
    int i = 0;
    while(i < n)
    {
        i = i + 1;
    }
    return i;
}
```

Loop can not be addressed by first-order logic.

Can we tell computer this is a loop?

1.1 Propositional Logic

Step 1. Convert it into mathematical formula.

Step 2. Ask the computer to solve the formula.

1.2 First Order Logic

Step 1. Convert it into first logic formula.

Step 2. Ask the computer to solve the formula.

Part I: Computer Aided Reasoning

```
int clone(unsigned int n)
{
    int i = 0;
    while(i < n)
    {
        i = i + 1;
    }
    return i;
}
```

➡

```
method clone(n: nat) returns (b : nat)
  ensures b == n
{
   var i := 0;
   while i < n
      invariant 0 <= i
   {
      i := i + 1;
   }
   return i;
}
```

1.1 Propositional Logic

Step 1. Convert it into mathematical formula.

Step 2. Ask the computer to solve the formula.

1.2 First Order Logic

Step 1. Convert it into first logic formula.

Step 2. Ask the computer to solve the formula.

1.3 Auto-active Proof

Step 1. Axiom system

Step 2. Ask the computer to check the invariants

Part I: Computer Aided Reasoning

```
int clone(unsigned int n)
{
    int i = 0;
    while(i < n)
    {
        i = i + 1;
    }
    return i;
}
```

➡

```
method clone(n: nat) returns (b : nat)
  ensures b == n
{
   var i := 0;
   while i < n
      invariant 0 <= i
   {
      i := i + 1;
   }
   return i;
}
```

		Description	Line	Column
⊗	1	A postcondition might not hold on this return path.	10	3
	2	This is the postcondition that might not hold.	2	12

Part I: Computer Aided Reasoning

```c
int clone(unsigned int n)
{
    int i = 0;
    while(i < n)
    {
        i = i + 1;
    }
    return i;
}
```

➡

```dafny
method clone(n: nat) returns (b : nat)
  ensures b == n
{
   var i := 0;
   while i < n
      invariant 0 <= i && i <= n
   {
      i := i + 1;
   }
   return i;
}
```

```
Dafny 2.3.0.10506

Dafny program verifier finished with 1 verified, 0 errors
Program compiled successfully
```

Part I: Computer Aided Reasoning Overview

1.1 Propositional Logic

Step 0. Proof.

Step 1. Convert program into mathematical formula.

Step 2. Ask the computer to solve the formula.

1.2 First Order Logic

Step 1. Convert program into first logic formula.

Step 2. Ask the computer to solve the formula.

1.3 Auto-active Proof

Step 1. Axiom system

Step 2. Ask the computer to check the invariants

TA
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3 Labs in total.
Good luck to all!